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TESE DE DOUTORAMENTO

Superstring Sigma Model Computations Using the Pure Spinor Formalism

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Resumo

Nesta tese são apresentadas duas aplicações do modelo sigma para a supercorda usando o formalismo de espinores puros. A primera aplicação é o cálculo da invariância conforme a um-loop para a supercorda tipo II, resultando em equações de movimento no super-espaço para campos de fundo acoplados com a supercorda. A segunda aplicação está relacionada com a invariância BRST da supercorda heterótica no nível quântico, que permite encontrar correções oriundas da teoria de supercordas para os vínculos de super Yang-Mills/supergravidade em dez dimensões.

Palavras Chaves: Supersimetria; Supercordas; Modelo Sigma; Correçẽs de Chern-Simons.

Áreas do conhecimento: Física de Partículas e Campos.

Abstract

In this thesis are presented two aplications of the sigma model for the superstring in the pure spinor formulation. The first aplication concerns the computation of the one-loop conformal invariance for the type II superstring, resulting in equations of motion written in superspace for the background fields coupled to the superstring. The second application is related to the BRST invariance of the heterotic superstring at the quantum level, which allows to find stringy corrections to the ten-dimensional super Yang-Mills/supergravity constraints.

Key Words: Supersymmetry; Superstrings; Sigma Model; Chern-Simons Corrections.

Areas of Knowledge: Fields and Particles Physics.

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Chapter 1

Introduction

The description of Physics in terms of fields dates back to the 19th century and had as origin the study of the electric and magnetic phenomena. Since then, the field language has seem appropriate to describe electromagnetism, gravitation, and the remaining two type of interaction discovered in the 20th century; namely the weak and strong interactions. The Standard Model of particle physics, which describes all but gravitational phenomena, is a beautiful example of a *unified* description for various fundamental interactions in terms of quantum fields. Nevertheless the Standard Model can be thought of as a built theory, which can be adjusted if some minor changes are required by the experiments. Furthermore, there are ingredients, in the philosophy of constructing, that are put by hand instead of deduced from more fundamental principles, for example, the way various particles are acomodated in the standard model multiplets. In some way, the ability to adjust such a theory also leaves the unsatisfactory taste of not having the right core from where to extract it in a *unique* manner.

Although the gravitational field has a well established classical field description, its quantum description has been elusive for quite long, as well as its incorporation, together with the other three interactions, in a single framework. Perhaps this first fact is an indication that the right type of description has not been used.

An important step in the direction of a quantum theory of gravity has been provided by precisely changing the type of description used in particle physics, namely Quantum Field Theory. String theory, which was *accidentally* discovered by studying an apparently singular behavior of the mass and the spin of some heavy particles in the late sixties; is a different proposal for describing particle physics. A string is a one-dimensional object, which expand a two-dimensional surface as it evolves in time, called the *worldsheet*. In its simplest version, namely the bosonic string, the

spectrum of particles is obtained by quantizing the modes of vibration of closed and open strings. In the first case, the massless sector of the spectrum contains a particle of spin two and mass zero, which is the graviton. In the second case, the massless particle of the spectrum, which has spin one is the photon. In this simplified model one can already handle with gravity and electromagnetism using a single framework. Actually this is not the first time that a single framework contains gravity and a gauge field. This is the case of the Kaluza-Klein theories, which appear as a compactification of a five-dimensional gravity theory to four dimensions. In string theory the appearance of extra dimensions is "natural", as explained below. In that sense, string theory also has room for Kaluza-Klein theories.

String theory is a huge subject of study and it is not the aim of this thesis to continue discussing their generalities. So in the following a description more focused in the topic of interest will be given.

1.1 Strings in a Generic Background

It is known since the early eighties that the coupling of strings to a generic backgrounds puts restriction on it, namely, puts the background on-shell. This equations of motion for the background can be computed perturbatively by considering the quantum regime of the worldsheet symmetries. In this section it will be discussed the bosonic string and superstring in a generic background.

1.1.1 Bosonic String Sigma Model

In the simplest case, a bosonic string propagates in a Minkowski space-time. In such a case, the theory possess conformal symmetry at the worldsheet level. However, of primordial interest in this thesis is to consider the case when the strings propagates in a curved space-time, which is described by coupling the bosonic string to a generic space-time metric. Such a coupling is described by a non-linear sigma model action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \partial_a X^m \partial_b X^n G_{mn}(X), \tag{1.1}$$

where X^m describe the coordinates of the string in D dimensional space-time, g_{ab} is a metric for the worldsheet, α' is proportional to the inverse of the string tension and G_{mn} is the space-time metric. This action is a direct generalization of the Polyakov action [1] when the Minkowski $\eta_{mn} = diag(-1, 1, \ldots 1)$ metric is replaced by the Riemannian metric. The interest of studying this type of action is related with the information one can extract out of it. As well as the preservation of the conformal

symmetry at the quantum level indicates that space-time should be 26 dimensional in the Minkowski space case, in the curved space case the preservation of this symmetry at the quantum level makes the space-time metric to satisfy the Einstein equations [2] [3], as will reviewed in detail. This is a way to obtain equations of motion for space-time fields, which could help to know the structure of the string effective action. Furthermore, perturbative methods can be used to compute stringy corrections to space-time equations of motion, giving also hints of string corrected effective actions: this requirement of conformal invariance can be computed perturbatively in the string parameter α', so the Einstein equations can receive stringy corrections.

The space-time metric is associated to one of the massless bosonic string states, namely the graviton. There are two more states at the massless level which can be associated to an antisymmetric tensor, denoted by B_{mn} and a scalar field Φ known as dilaton. In this way a generalized sigma model can be constructed, whose action in the conformal gauge is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (\sqrt{g}g^{ab}\partial_a X^m \partial_b X^n G_{mn} + \epsilon^{ab}\partial_a X^m \partial_b X^n B_{mn}) + \frac{1}{2\pi} \int d^2\sigma \sqrt{g}r^{(2)}\Phi(X),$$
(1.2)

where ϵ^{ab} is the purely antisymmetric tensor in two dimensions and $r^{(2)}$ is the scalar curvature in two dimensions. The requirement of conformal invariance at the quantum level also puts the background field on-shell. These equations of motion can be found as requirements for scale invariance, i.e. computing the beta function for the generalized non-linear sigma model (1.2).

The condition for conformal invariance can be written as conditions for the stress-energy tensor being traceless:

$$\langle T_a{}^a \rangle = \beta^G_{mn}g^{ab}\partial_a X^m \partial_b X^n + \beta^B_{mn}\epsilon^{ab}\partial_a X^m \partial_b X^n + \beta^\Phi R^{(2)},$$
(1.3)

where

$$\beta^G_{mn} = R_{mn} - \frac{1}{4}H_{mlr}H_n{}^{lr} + 2\nabla_m\nabla_n\Phi,$$
(1.4)

$$\beta^B_{mn} = -\frac{1}{2}\nabla^r H_{mnr} + \nabla_l\Phi H^l{}_{mn},$$
(1.5)

$$\beta^\Phi = -\frac{D-26}{12} + \alpha'\left(R - \frac{H^2}{12} + 4\nabla^2\Phi - 4(\nabla\Phi)^2\right),$$
(1.6)

and H_{mnp} are the components of the three form $H = dB$. So, the theory is conformal invariant if the beta-functions are zero. In the following it will be explained the procedure to compute this β-functions, taking [4] as reference.

Covariant Background Field Expansion

By making a perturbative expansion it will be found a diagramatic expression for the terms in such expansion. With this goal, it will be introduced the partition function

$$Z_J \equiv e^{-W[J]} = \int [dX] exp\left(-(S[X] + X \cdot J)\right), \tag{1.7}$$

which defines the functional generator $W[J]$ of connected diagrams, where

$$X \cdot J = \int d^2\sigma X^m J_m. \tag{1.8}$$

Variating respect of J, one defines the mean field

$$X_0^m \equiv \frac{\delta W}{\delta J_m} = \frac{1}{Z_J} \int [dX] X^m e^{S_J}, \tag{1.9}$$

with

$$S_J \equiv S[X] + X \cdot J. \tag{1.10}$$

This mean field X_0^m will play the role of a background field; it will be the field around which the perturbative expansion will be made. The effective action is defined by

$$\Gamma \equiv W - X_0 \cdot J. \tag{1.11}$$

From this equation, the current can be written as

$$J_m = -\frac{\delta\Gamma}{\delta X_0^m}, \tag{1.12}$$

so, the effective action takes the form

$$\Gamma = W + X_0 \cdot \frac{\delta\Gamma}{\delta X_0}, \tag{1.13}$$

what allows to write $exp(-\Gamma)$ using (1.7), (1.12) and (1.13):

$$e^{-\Gamma}[X_0] = \int [dY] \exp\left(-(S[X_0 + Y] - Y \cdot \frac{\delta\Gamma}{\delta X_0})\right), \tag{1.14}$$

where $Y^m \equiv X^m - X_0^m$. The field Y^m will play the role of a quantum field in the background field method. Insted of using the last functional, it will be used

$$\Omega[X_0] = \int [dY] \exp\left(-(S[X_0 + Y] - S[X_0] - Y \cdot \frac{\delta\Gamma}{\delta X_0})\right). \tag{1.15}$$

This will be the generator of the 1PI diagrams. Subtracting $S[X_0]$ from the exponential in (1.14), the expansions of the fields around X_0 will always contain the

quantum field Y^m. Making a Taylor expansion around X_0 a power series in Y^m will be obtained. Each term of such a expansion will not be invariant under general coordinate transformations of spacetime, since Y^m is a subtraction of two coordinates, does not have such an invariance. That is why it will be useful to find a system of coordinates in which the coordinate invariance is manifest, such a coordinate system is denoted as normal coordinate system.

Normal Coordinate System Let X_0^m be coordinates for a point P_0 in space-time and $X_0^m + Y^m$ coordinates for a point P, it is possivel to find another coordinate system by using a geodesic that joins both points. By considering a parameter t for the geodesic

$$\frac{d^2\lambda^m}{dt^2} + \Gamma^m_{np}\frac{d\lambda^n}{dt}\frac{\lambda^p}{dt} = 0 \tag{1.16}$$

such that $\lambda^m(0) = X_0^m$ and $\lambda(1) = X_0^m + Y^m$. Defining ξ^m as

$$\xi^m \equiv \frac{d\lambda^m}{dt}(0), \tag{1.17}$$

it will be a tangent vector to the geodesic in P_0, and as such, will transform as a vector under a change of coordinates. So, any geometrical object when expanded in Taylor series around X_0^m will be a diffeomorphism expression.

$$T_{m_1...m_i}(X_0 + \xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial}{\partial\xi^{n_1}}...\frac{\partial}{\partial\xi^{n_k}}\right)T_{m_1...m_i}(X_0)\xi^{n_1}...\xi^{n_k}. \tag{1.18}$$

Supposing a solution for the geodesic equation in Taylor series (1.16)

$$\lambda^m(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k}{dt^k}\lambda^m(t)t^k, \tag{1.19}$$

then this solution, with the initial conditions already given has the form

$$\lambda^m(t) = X_0^m + \xi^m t - \frac{1}{2}\Gamma^m_{n_1n_2}\xi^{n_1}\xi^{n_2}t^2 - \frac{1}{3!}\Gamma^m_{n_1n_2n_3}\xi^{n_1}\xi^{n_2}\xi^{n_3}t^3 - ..., \tag{1.20}$$

$$= X_0^m + \xi^m t - \sum_{k=2}^{\infty} \frac{1}{k!}\Gamma^m_{n_1...n_k}\xi^{n_1}...\xi^{n_k}t^k, \tag{1.21}$$

where

$$\Gamma^m_{n_1n_2n_3} = \partial_{n_1}\Gamma^m_{n_2n_3} - \Gamma^l_{n_1n_2}\Gamma^m_{ln_3} - \Gamma^l_{n_1n_3}\Gamma^m_{n_2l}, \tag{1.22}$$

$$\equiv \nabla_{n_1}\Gamma^m_{n_2n_3}. \tag{1.23}$$

The definition (1.23) is used recursively for defining $\Gamma^m_{n_1...n_i}$ as the covariant derivative acting only in the lower indices

$$\Gamma^m_{n_1...n_i} \equiv \nabla_{n_1} \Gamma^m_{n_2...n_i}. \tag{1.24}$$

Finally, $\lambda(t = 1)$ defines a coordinate transformation in which Y^m is written as a contravariant vector in this coordinate system

$$\bar{\Gamma}^m_{(n_1...n_i)} = 0, \tag{1.25}$$

this defines the *normal coordinate system*. Here the bar notation indicates that the expression is valid in a normal coordinate system.

Using such a coordinate system several expressions are simplified. Christoffel symbols cancel, although not their derivatives, so the curvature tensor in components is written as

$$\bar{R}^m{}_{nlp} = \partial_l \bar{\Gamma}^m_{np} - \partial_p \bar{\Gamma}^m_{nl}. \tag{1.26}$$

From (1.25) when $i = 3$

$$\partial_n \bar{\Gamma}^m_{lp} = -\partial_l \bar{\Gamma}^m_{pn} - \partial_p \bar{\Gamma}^m_{ln}. \tag{1.27}$$

Adding $2\partial_n \bar{\Gamma}^m_{pl}$ to both sides of (1.27) and using the symmetry of the Christoffel symbols, one gets

$$\partial_{n_1} \bar{\Gamma}^m_{n_2 n_3} = \frac{1}{3} \left(\bar{R}^m{}_{n_2 n_1 n_3} + \bar{R}^m{}_{n_3 n_1 n_2} \right). \tag{1.28}$$

In the following this result will be used to find an expansion in normal coordinates for G_{mn}.

Perturbative Expansion

From (1.18) it can be found

$$\begin{aligned} \bar{G}_{mn}(X_0 + \xi) &= \bar{G}_{mn}(X_0) + \frac{\partial}{\partial \xi^l} \bar{G}_{mn}(X_0)\xi^l + \\ &\quad + \frac{1}{2} \frac{\partial^2}{\partial \xi^l \partial \xi^p} \bar{G}_{mn}(X_0)\xi^l \xi^p.... \end{aligned} \tag{1.29}$$

As this expression is written in normal coordinates, then

$$\frac{\partial}{\partial \xi^l} \bar{G}_{mn} = \nabla_l \bar{G}_{mn}, \tag{1.30}$$

but if the metric G_{mn} is covariantly constant, then this term does not appear in the expansion.

The derivatives of the third term in (1.29) can be written as

$$\partial_l \partial_p \bar{G}_{mn} = \nabla_l \nabla_p \bar{G}_{mn} + \partial_l \bar{\Gamma}^q_{pm} \bar{G}_{qn} + \partial_l \bar{\Gamma}^q_{pn} \bar{G}_{mq}. \tag{1.31}$$

Symmetryzing (1.31) in the indices l and p, and using (1.28)

$$\partial_{(l} \partial_{p)} \bar{G}_{mn} = \nabla_{(l} \nabla_{p)} \bar{G}_{mn} + \frac{1}{3} [\bar{R}_{n(pl)m} + \bar{R}_{nm(lp)} + \bar{R}_{m(pl)n} + \bar{R}_{mn(lp)}]. \tag{1.32}$$

Replacing this expression in (1.29), and using the symmetries of the curvatures tensor

$$R_{mnlp} = -R_{mnpl}, \quad R_{mnlp} = R_{lpmn}, \tag{1.33}$$

one obtains

$$\bar{G}_{mn}(X_0 + Y) = \bar{G}_{mn}(X_0) + \frac{1}{3} \bar{R}_{mlpn} \xi^l \xi^p + ..., \tag{1.34}$$

where again, the covariant derivatives of G_{mn} are zero. Now, the expansion (1.34) is written purely in tensorial terms and as such is valid in any coordinate system. So, the bar notation is no longer necessary. It will also be necessary to compute $\partial_a(X_0^m + Y^m)$. With this aim, one computes (1.20) in $t = 1$ and take derivatives :

$$\partial_a(X_0^m + Y^m) = \partial_a X_0^m + \partial_a \xi^m - \frac{1}{2} \partial_n \bar{\Gamma}^m_{lp} \xi^l \xi^p \partial_a X_0^n - ..., \tag{1.35}$$

and using again (1.28),

$$\partial_a(X_0^m + Y^m) = \partial_a X_0^m + \nabla_a \xi^m + \frac{1}{3} \bar{R}^m{}_{lpn} \partial_a X_0^n \xi^l \xi^p - ..., \tag{1.36}$$

where

$$\nabla_a \xi^m \equiv \partial_a \xi^m + \Gamma^m_{lp} \xi^l \partial_a X_0^p \tag{1.37}$$

is the covariantization of $\partial_a \xi^m$. Then, (1.36) can be written manifestly covariant. Finally multiplying (1.34) and (1.36) one finds

$$S_G[X_0 + Y] = S_G[X_0] + \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{|g|} g^{ab} G_{mn} \partial_a X_0^m \nabla_b \xi^n \tag{1.38}$$

$$+ \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{|g|} g^{ab} \left(G_{mn} \nabla_a \xi^m \nabla_b \xi^n + R_{mlpn} \partial_a X_0^m \partial_b X_0^n \xi^l \xi^p \right) + ...,$$

where Σ denotes the two-dimensional manifold. In subsequent chapters this notation will be dropped off. Now, for the antisymmetric tensor B_{mn} one obtains

$$
\begin{aligned}
B_{mn}(X_0 + Y) &= B_{mn}(X_0) + \nabla_p B_{mn}(X_0)\xi^p + \frac{1}{2}(\nabla_p \nabla_q B_{mn}(X_0) \\
&\quad + \frac{1}{3}R^l{}_{pqm}B_{ln}(X_0) + \frac{1}{3}R^l{}_{pqn}B_{ml}(X_0))\xi^p\xi^q + ... \quad (1.39)
\end{aligned}
$$

From (1.39) and (1.36), it can be found

$$
\begin{aligned}
S_B[X_0 + Y] &= S_B[X_0] + \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{ab}(B_{mn}(X_0)\partial_a X_0^m \nabla_b \xi^n \\
&\quad + \frac{1}{2}\nabla_l B_{mn}\partial_a X_0^m \partial_b X_0^n \xi^l) \quad (1.40) \\
&\quad + \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{ab}[B_{mn}(X_0)\nabla_a \xi^m \nabla_b \xi^n + 2\nabla_l B_{mn}\partial_a X_0^m \xi^l \nabla_b \xi^n \\
&\quad + \frac{1}{2}(\nabla_l \nabla_p B_{mn} + 2B_{mq}R^q{}_{lpn})\partial_a X_0^m \partial_b X_0^n \xi^l \xi^p] + ...
\end{aligned}
$$

Nevertheless, note that the action for the antisymmetric tensor S_B is invariant under the following transformations

$$
B_{mn} \rightarrow B_{mn} + \partial_m \Lambda_n - \partial_n \Lambda_m, \quad (1.41)
$$

form some vector $\Lambda_m(X)$. So it is important that the terms in the expansion have also this symmetry, and one look for a expansion in terms of the field strength

$$
H_{mnl} \equiv \nabla_m B_{nl} + \nabla_n B_{lm} + \nabla_l B_{mn} = \nabla_{[m}B_{nl]}, \quad (1.42)
$$

which is invariant under (1.41). The square bracket notation in sub-indices means they are antisymmetrized. Up to surface terms, one finds the following expression at second order in ξ

$$
S_B[X_0+Y] = S_B[X_0] + \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{ab}\left(H_{lmn}\partial_a X_0^l \xi^m \nabla_b \xi^n + \frac{1}{2}\nabla_l H_{mnp}\partial_a X_0^m \partial_b X_0^n \xi^l \xi^p\right) + ...
$$
$$
(1.43)
$$

The expansion for S_Φ can easily be obtained

$$
\begin{aligned}
S_\Phi[X_0 + Y] &= S_\Phi[X_0] + \frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{|g|}R^{(2)}\nabla_m \Phi(X_0)\xi^m \\
&\quad + \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{|g|}R^{(2)}\nabla_m \nabla_n \Phi(X_0)\xi^m \xi^n + ... \quad (1.44)
\end{aligned}
$$

If X_0 satisfy its classical equation of motion, then the linear term in ξ vanishes. To read the propagators, it will be useful to implement an orthogonal frame, or vielbeins, for which $G_{mn} = e_m{}^i e_n{}^j \eta_{ij}$.

Orthogonal Frame Denoting by $\{\hat{\partial}_m\}$ the vectors in a coordinate basis, another basis can be written as a linear combination of them

$$\hat{e}_i = e_i{}^m \hat{\partial}_m, \tag{1.45}$$

with $i = 0, ..., D - 1$. For being orthonormal, it must satisfy

$$G(\hat{e}_i, \hat{e}_j) = G_{mn} dX^m \otimes dX^n (e_i{}^l \hat{\partial}_l, e_j{}^p \hat{\partial}_p) = \eta_{ij}, \tag{1.46}$$

then

$$G_{mn} e_i{}^m e_j{}^n = \eta_{ij}. \tag{1.47}$$

Denoting by $e_m{}^i$ the inverse elements of $e_i{}^m$, they satisfy

$$e_i{}^m e_m{}^j = \delta_i{}^j, \qquad e_m{}^i e_i{}^n = \delta_m{}^n, \tag{1.48}$$

and from (1.47) one finds

$$\eta_{ij} e_m{}^i e_n{}^j = G_{mn}. \tag{1.49}$$

In this base, a conexion ω_m is introduced, with components $\omega_m{}^i{}_j$. It is defined by the condition that the covariant derivative acting in the tetrad base is zero:

$$\nabla_m e_n{}^i \equiv \partial_m e_n{}^i - \Gamma^l_{mn} e_l{}^i + \omega_m{}^i{}_j e_n{}^j = 0. \tag{1.50}$$

Now, it is easy to see how the components of a vector are related among the two basis

$$\xi = \xi^i \hat{e}_i = \xi^i e_i{}^m \hat{\partial}_m = \xi^m \hat{\partial}_m, \tag{1.51}$$

from which

$$\xi^m = \xi^i e_i{}^m. \tag{1.52}$$

Therefore, using (1.47), (1.50) and (1.52)

$$G_{mn} \nabla_a \xi^m \nabla_b \xi^n = \delta_{ij} e_m{}^i e_n{}^j \nabla_a \xi^m \nabla_b \xi^n, \tag{1.53}$$

$$= (\nabla_a \xi)^i (\nabla_b \xi)^i. \tag{1.54}$$

In this case, ∇_a denote that the derivative ∂_a has been covariantized, given by

$$(\nabla_a \xi)^i = \partial_a \xi^i + \omega_m{}^i{}_j \partial_a X^m \xi^j, \tag{1.55}$$

and it is assumed that the derivative acting in $e_m{}^i$ is zero.

With the help of the tetrad base, the following expansion for the generalized bosonic non-linear sigma model is found

$$
\begin{aligned}
S_\sigma[X_0 + Y] \;=\; & S_\sigma[X_0] + \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{|g|} g^{ab}[(\nabla_a\xi)^i(\nabla_b\xi)^i \\
& + R_{mijn}(X_0)\partial_a X_0^m \partial_b X_0^n \xi^i \xi^j] \\
& + \frac{1}{3\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{|g|} g^{ab} R_{mijk}(X_0)\partial_a X_0^m (\nabla_b\xi)^k \xi^i \xi^j \\
& + \frac{1}{12\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{|g|} g^{ab} R_{iklj}(X_0)(\nabla_a\xi)^i(\nabla_b\xi)^j \xi^k \xi^l \quad (1.56) \\
& + \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \epsilon^{ab}[H_{mij}(X_0)\partial_a X_0^m \xi^i(\nabla_b\xi)^j \\
& + \frac{1}{2}\nabla_i H_{mnj}(X_0)\partial_a X_0^m \partial_b X_0^n \xi^i \xi^j] \\
& + \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{|g|} R^{(2)} \nabla_i \nabla_j \Phi(X_0)\xi^i \xi^j + \dots.
\end{aligned}
$$

One-loop conformal invariance

In this section it will be studied the conditions for the energy-momentum tensor being traceless. Noting that the functional generator

$$
\Omega = e^{-\Gamma}., \tag{1.57}
$$

depends only on the worldsheet metric (in a fixed gauge) and, as a consequence of the diffeomorphism invariance satisfies

$$
0 = \int_\Sigma d^2\sigma \frac{\delta\Gamma}{\delta g^{ab}}(\nabla^a v^b + \nabla^b v^a), \tag{1.58}
$$

which in the conformal gauge and using coordinates $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$ takes the form

$$
0 = \int_\Sigma d^2 z \left(\frac{\delta\Gamma}{\delta g^{zz}}\nabla^z v^z + \frac{\delta\Gamma}{\delta g^{\bar{z}\bar{z}}}\nabla^{\bar{z}} v^{\bar{z}} - \frac{1}{2}\frac{\delta\Gamma}{\delta\omega}(\nabla_z v^z + \nabla_{\bar{z}} v^{\bar{z}}) \right), \tag{1.59}
$$

where ω is a conformal factor. Integrating by parts this expression

$$
\begin{aligned}
0 = \int_\Sigma d^2 z \sqrt{\hat{g}} \quad & \left[\left(\nabla_z \left(\frac{1}{2\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta\omega} \right) - \nabla^z \left(\frac{1}{\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta g^{zz}} \right) \right) v^z \\
& + \left(\nabla_{\bar{z}} \left(\frac{1}{2\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta\omega} \right) - \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta g^{\bar{z}\bar{z}}} \right) \right) v^{\bar{z}} \right], \quad (1.60)
\end{aligned}
$$

and as v^z and $v^{\bar{z}}$ are arbitrary, the following set of equations is obtained

$$\nabla_z \left(\frac{1}{2\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta \omega} \right) = \nabla^z \left(\frac{1}{\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta g^{zz}} \right) \tag{1.61}$$

$$\nabla_{\bar{z}} \left(\frac{1}{2\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta \omega} \right) = \nabla^{\bar{z}} \left(\frac{1}{\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta g^{\bar{z}\bar{z}}} \right). \tag{1.62}$$

These equations are the analog of the conservation of the classical energy-momentum tensor.

It is possible to show that the right hand side are derivatives of the expectation value of the zz and $\bar{z}\bar{z}$ components of this tensor, as explained in the following From (1.57),

$$\frac{\delta \Gamma}{\delta g^{zz}} = -\frac{1}{\Omega} \frac{\delta \Omega}{\delta g^{zz}}. \tag{1.63}$$

The measure element is chosen in such a way for not contributing to the energy-momentum tensor. This allows to write

$$\nabla_z \left(\frac{4\pi}{\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta g^{zz}} \right) = \nabla^z \langle T_{zz} \rangle, \tag{1.64}$$

and in the same way

$$\nabla_{\bar{z}} \left(\frac{4\pi}{\sqrt{\hat{g}}} \frac{\delta \Gamma}{\delta g^{\bar{z}\bar{z}}} \right) = \nabla^{\bar{z}} \langle T_{\bar{z}\bar{z}} \rangle. \tag{1.65}$$

The left hand side of (1.61) will be identified with the expectation value of the component $z\bar{z}$ of the energy-momentum.

In this way, independent of the metric g component under consideration, the variation of the effective respect of the metric is identified with the energy-momentum tensor in the quantum regime:

$$\langle T_{ab} \rangle = \frac{4\pi}{\sqrt{|g|}} \frac{\delta \Gamma}{g_{ab}}, \tag{1.66}$$

therefore, (1.61) takes the form of a conservation law

$$\nabla_z \langle T_{\bar{z}z} \rangle + \nabla_{\bar{z}} \langle T_{zz} \rangle = 0. \tag{1.67}$$

The idea is to use the value of $\langle T_{zz} \rangle$ computed at $1-$loop, to compute the trace of the energy-momentum tensor. Initially can be considered a flat worldsheet, and as discussed later, worldsheet curvature effects will be taken into account. Using the notation $q = q^z$, $\bar{q} = q^{\bar{z}}$, in momenta space the conservation law (1.67) takes the form

$$\bar{q}\langle T_{z\bar{z}}(q)\rangle + q\langle T_{zz}(q)\rangle = 0. \tag{1.68}$$

It will be first computed the contribution to $\langle T_{zz}\rangle$ coming from the variation of the effective action containing the term S_G. From (1.66) and (1.63)

$$\langle T_{zz}\rangle = \frac{1}{\Omega[X_0]}\frac{4\pi}{\sqrt{|g|}}\int [d\xi]exp\{-(S[X_0+\xi]-S[X_0])\}\frac{\delta(S_G[X_0+\xi]-S_G[X_0])}{\delta g^{zz}}. \tag{1.69}$$

The term in the exponential can be written as

$$S_G[X_0+\xi]-S_G[X_0] = S_{Free}+S_{Int}, \tag{1.70}$$

with

$$S_{Free} = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{|g|}g^{ab}\partial_a\xi^i\partial_b\xi^i \tag{1.71}$$

and

$$S_{Int} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{|g|}g^{ab}R_{imjn}\partial_a X_0^m\partial_b X_0^n\xi^i\xi^j. \tag{1.72}$$

Having chosen a tetrad basis allows to find an expression for the propagator, which can be expressed diagrammatically as

$$\text{i} \xrightarrow{\quad q \quad} \text{j} = \frac{4\pi\alpha'\eta^{ij}}{|q|^2} \tag{1.73}$$

while from S_{Int} can be found a vertex

$$\text{(vertex diagram)} = -\frac{1}{4\pi\alpha'}\sqrt{|g|}g^{ab}R_{imjn}\partial_a X_0^m\partial_a X_0^n\xi^i\xi^j. \tag{1.74}$$

where the curved lines represent background fields. Writing (1.69) as

$$\langle T_{zz}\rangle = \frac{1}{\Omega[X_0]}\int [d\xi]e^{-(S_{Liv})}\left(\frac{1}{\alpha'}\partial_z\xi^i\partial_z\xi^i + ...\right)e^{-S_{Int}} \tag{1.75}$$

and making an expansion of the exponential, the following diagram can be formed

$$\frac{1}{\alpha'}\partial_z\xi^i\partial_z\xi^j \quad\times \qquad\qquad -\frac{1}{4\pi\alpha'}\sqrt{|g|}g^{ab}R_{jmkn}\partial_aX_0^m\partial_bX_0^n\xi^j\xi^k$$

$$\tag{1.76}$$

where the cross represents an insertion of the energy-momentum tensor. This diagram leads to

$$\langle T_{zz}\rangle_G = \frac{1}{4}\int\frac{d^2l}{2\pi}\frac{\bar{l}(\bar{l}+\bar{q})}{l^2(l+q)^2}\{R_{mn}\partial_aX_0^m\partial^aX_0^n\}(q). \tag{1.77}$$

In this equation the keys denote an expression in momentum space, but independent of the momenta l.

To compute this integral in (1.77), one can use a Feynman parameter x to write it as a known integral, whose value could be found using the dimensional regularization formulas [5]. Introducing the parameter x, the integrand of (1.77) is written as

$$\frac{\bar{l}(\bar{l}+\bar{q})}{l^2(l+q)^2} = \int_0^1 dx\frac{(l^1-il^2)^2+(l^1-il^2)(q^1-iq^2)}{[xl^2+(1-x)(l+q)^2]^2}. \tag{1.78}$$

Defining

$$k^a \equiv l^a + (1-x)q^a, \quad a = 1,2 \tag{1.79}$$

then

$$\int\frac{dld\bar{l}}{2\pi}\frac{\bar{l}(\bar{l}+\bar{q})}{l^2(l+q)^2} = \int_0^1 dx\int\frac{d^2k}{\pi}\frac{(k^1)^2-(k^2)^2-2ik^1k^2-x(1-x)(q^1-iq^2)^2}{(k^2+\Delta)^2}, \tag{1.80}$$

with

$$\Delta \equiv x(1-x)q^2. \tag{1.81}$$

In the integral (1.80) there are omitted linear terms in k that vanishes because of symmetry. Extending the two-dimensional space to $d = 2 + \epsilon$, (1.80) has a known form. The quadratic terms in k cancel among them using

$$\int d^dk\frac{k^ak^b}{(k^2+\Delta)^2} = \frac{\pi^{\frac{d}{2}}\delta^{ab}}{2}\Gamma(1-\frac{d}{2})\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}. \tag{1.82}$$

Now, using

$$\int d^d k \frac{1}{(k^2 + \Delta)^2} = \pi^{\frac{d}{2}} \Gamma(2 - \frac{d}{2}) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \tag{1.83}$$

in the limit $\epsilon \to 0$, one obtains
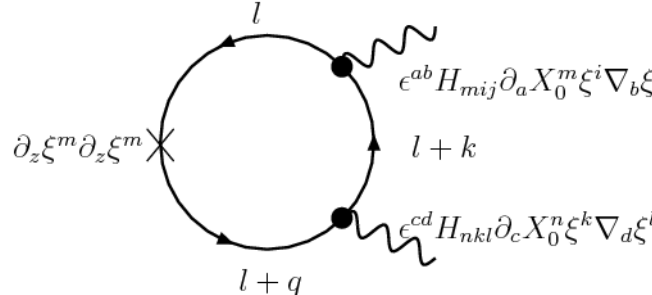
$$\int \frac{d^2 l}{2\pi} \frac{\bar{l}(\bar{l} + \bar{q})}{l^2(l + q)^2} = -\frac{\bar{q}}{q}. \tag{1.84}$$

Substituting (1.84) in (1.68) one obtains an expression for the trace of the energy momentum tensor

$$\langle T_{z\bar{z}} \rangle_G = \frac{1}{4} R_{mn} \partial_a X_0^m \partial^a X_0^n, \tag{1.85}$$

which was the desired result.

It can also be found an interaction with the first term in (1.43). A diagram with only the linear term in the expansion of the exponential is canceled by the antisymmetry of H, therefore, such a term should be considered in quadratic order. The type of interaction is given by

$$\langle T_{zz} \rangle_{H_2} = -\frac{1}{8\alpha'} \frac{(4\pi\alpha')^3}{(4\pi\alpha')^2} \frac{\epsilon^{ab}}{\sqrt{|g|}} \frac{\epsilon^{cd}}{\sqrt{|g|}} \partial_a X_0^m \partial_c X_0^n H_{mij} H_{nlk} \eta^{il} \eta^{jk} \tag{1.87}$$

$$\times \int \frac{dl d\bar{l}}{2(2\pi)^2} \frac{(l + q)_z (l + k)_b l_z l_d}{l^2(l + q)^2(l + k)^2}$$

Without lost of generality, one can make $k = 0$. Using

$$l_b l_d = \frac{1}{2} g_{bd} l^2 \tag{1.88}$$

and

$$\epsilon^{ab} \epsilon^{cd} g_{bd} = g^{ac}, \tag{1.89}$$

the equation (1.87) is written

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$$\langle T_{zz}\rangle_{H_2} = -\frac{1}{64}H^2_{mn}g^{ab}\partial_a X^m_0 \partial_b X^n_0 \int \frac{dl d\bar{l}}{2\pi}\frac{\bar{l}(\bar{l}+\bar{q})}{l^2(l+q)^2}, \qquad (1.90)$$

with

$$H^2_{mn} \equiv H_{mlp}H_n{}^{lp}. \qquad (1.91)$$

Using the computed value for this integral (1.84) and having in mind the existence of other four identical configurations to the diagram (ref diagram 2) the following expression for $\langle T_{zz}\rangle$ can be found

$$\langle T_{zz}\rangle_{H_2} = \frac{1}{16}\frac{\bar{q}}{q}H^2_{mn}g^{ab}\partial_a X^m_0 \partial_b X^n_0. \qquad (1.92)$$

Finally, using the expression for the energy-momentum conservation (1.68) can be found the expression for $\langle T_{z\bar{z}}\rangle$.

Up to now the results are

$$\langle T_{z\bar{z}}\rangle = \left(\frac{1}{4}R_{mn} - \frac{1}{16}H_{mn}{}^2\right)\partial_a X^m_0 \partial^a X^n_0 + \left(-\frac{1}{8}\nabla^l H_{lmn}\right)\frac{\epsilon^{ab}}{\sqrt{|g|}}\partial_a X^m_0 \partial_b X^n_0 \quad (1.93)$$

Until now the contributions coming from the dilaton were ignored by choosing $g_{ab} = \delta_{ab}$. It seems that (1.93) would be all the contributions at first order in α'. Nevertheless, in a flat worldsheet, variations of the action including the dilaton respect of infinitesiaml variations of the metric, when evaluated in a flat worldsheet are different from zero. Making this variation

$$\delta S_\Phi = -\frac{1}{2\pi}\int_\Sigma d^2\sigma \delta(\sqrt{|g|}R^{(2)})\Phi(X), \qquad (1.94)$$

Palatini's identity can be used:

$$\delta R^{(2)}_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}, \qquad (1.95)$$

where

$$\delta\Gamma^c_{ab} \equiv -g^{cd}\delta g_{de}\Gamma^e_{ab} + \frac{1}{2}g^{cd}(\delta g_{da,b} + \delta g_{db,a} - \delta g_{ab,d}), \qquad (1.96)$$

to write

$$\sqrt{|g|}g^{ab}\delta R^{(2)}_{ab} = \nabla_b(\sqrt{|g|}g^{ab}\delta\Gamma^c_{ac}) - \nabla_c(\sqrt{|g|}g^{ab}\delta\Gamma^c_{ab}). \qquad (1.97)$$

Integrating by parts (1.94) one finds

$$\delta S_\Phi = \frac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{|g|} \{ g^{ab} \delta \Gamma^c_{ab} \partial_c \Phi(X) - g^{ab} \delta \Gamma^c_{ac} \partial_b \Phi(X) \}$$
$$+ \frac{1}{2\pi} \int_\Sigma d^2\sigma \{ \delta \sqrt{|g|} R^{(2)} \Phi(X) + \sqrt{|g|} \delta g^{ab} R^{(2)} \}. \qquad (1.98)$$

Replacing (1.96) in (1.98), integrating again by parts and making $g_{ab} = \delta_{ab}$, the energy-momentum tensor can be computed

$$T^{dil}_{ab} = 2(\partial_a \partial_b - \delta_{ab} \Delta) \Phi(X). \qquad (1.99)$$

The simbol Δ is the Laplacian in the worldsheet. In coordinates $(z, \bar{z})$,

$$T^{dil}_{z\bar{z}} = -\frac{1}{2} \Delta \Phi(X(\sigma)), \qquad (1.100)$$

and the trace is different from zero, as expected by the lack of conformal symmetry of S_Φ. To compute the contributions of this trace to the total conformal anomaly, one computes (1.100) in X_0,

$$\begin{aligned} \Delta \Phi(X_0) &= 2g^{z\bar{z}} \partial_z \partial_{\bar{z}} \Phi(X_0), \\ &= 2g^{z\bar{z}} \partial_z (\partial_m \Phi(X_0) \partial_{\bar{z}} X_0^m), \\ &= \partial_a X_0^n \partial^a X_0^m \partial_n \partial_m \Phi(X_0) + \Delta X_0^m \partial_m \Phi(X_0). \qquad (1.101) \end{aligned}$$

But as X_0 satisfy the classical equation of motion

$$\Delta X_0^m = -\Gamma^m_{nl} g^{ab} \partial_a X_0^n \partial^a X_0^l + \frac{1}{2} H^m{}_{nl} \frac{\epsilon^{ab}}{\sqrt{|g|}} \partial_a X_0^n \partial_b X_0^l, \qquad (1.102)$$

replacing (1.102) in (1.101), one finds

$$\Delta \Phi(X_0) = \nabla_m \nabla_n \Phi(X_0) \partial_a X_0^m \partial^a X_0^n + \frac{1}{2} \nabla^m \Phi(X_0) H_{mnl} \frac{\epsilon^{ab}}{\sqrt{|g|}} \partial_a X_0^n \partial_b X_0^l, \quad (1.103)$$

That is, the classical contributions coming from the variation of S_Φ are of the same order as the one-loop contributions coming from $\langle T_G \rangle$ and $\langle T_H \rangle$. Therefore, the following partial result can be written

$$\langle T_a^a \rangle = \beta^G_{mn} \partial_a X_0^m \partial^a X_0^n + \beta^B_{mn} \frac{\epsilon^{ab}}{\sqrt{|g|}} \partial_a X_0^m \partial_b X_0^n, \qquad (1.104)$$

with

$$\beta^G_{mn} = R_{mn}(X_0) - \frac{1}{4}H_{mlp}H_n{}^{lp}(X_0) + 2\nabla_m\nabla_n\Phi(X_0), \qquad (1.105)$$

$$\beta^B_{mn} = -\frac{1}{2}\nabla^l H_{lmn}(X_0) + H_{lmn}\nabla^l\Phi(X_0). \qquad (1.106)$$

To find the remaining terms in the beta functions, some computations of two point functions must be done, i.e two insertions of the energy momentum tensor. To see that this is the case, one must remember that because of the diffeomorphism symmetry in two dimensions, a worldsheet metric can be written as a scale factor times a flat metric. In this case, to state that a theory has Weyl symmetry is equivalent to say that the energy-momentum tensor is traceless, when computed using the metric $g_{ab} = e^{2\omega}\delta_{ab}$. This must be independent of the scale factor ω such that the result is valid for any curved worldsheet. Then, at least the first variation with respec to to ω must be zero. This first variation can be written as

$$\frac{\delta}{\delta\omega(z')}\langle T_{z\bar{z}}(z)\rangle_{e^{2\omega}\delta} = \frac{\delta\langle T_{z\bar{z}}(z)\rangle}{\delta g^{ab}}\frac{\delta g^{ab}}{\delta\omega}, \qquad (1.107)$$

and evaluating in $\omega = 0$, the variation (1.107) is written in terms of the two point function for the energy-momentum tensor

$$\frac{\delta}{\delta\omega(z')}\langle T_{z\bar{z}}(z)\rangle_{e^{2\omega}\delta} = -\frac{1}{\pi}\langle T_{z\bar{z}}(z)T_{z\bar{z}}(z')\rangle_\delta, \qquad (1.108)$$

where

$$\langle T_{z\bar{z}}(z)T_{z\bar{z}}(z')\rangle_\delta = \langle T^G_{z\bar{z}}(z)T^G_{z\bar{z}}(z')\rangle_\delta + 2\langle T^G_{z\bar{z}}(z)T^\Phi_{z\bar{z}}(z')\rangle_\delta + \langle T^\Phi_{z\bar{z}}(z)T^\Phi_{z\bar{z}}(z')\rangle_\delta. \qquad (1.109)$$

Integrating ω allows to write the trace of the energy-momentum tensor in terms of the results obtained from the computation of the two point function. The result will be something known: a term proportional to the worldsheet curvature scalar $R^{(2)}$, whose constant of proportionality contains the space-time dimension. Furthermore, some contributions fo the fields G and H will be found.

Consider for the time being just the terms in the classical action with Weyl symmetry: S_G and S_B. At the lowest order in α' in the two point function, $\langle T_{zz}T_{zz}\rangle$, the next graph is found

$$\partial_z\xi^i\partial_z\xi^i(\text{q}) \times \qquad \times \partial_z\xi^j\partial_z\xi^j(\text{-q})$$

with labels l and $q+l$ on the loop.

$$(1.110)$$

from which is obtained

$$\langle T_{zz}(q)T_{zz}(-q)\rangle = \frac{D}{8}\int d^2l\frac{\bar{l}^2(\bar{l}+\bar{q})^2}{l^2(l+q)^2}. \qquad (1.111)$$

To write the integral in (1.111) as an integral whose value is known, a Feynman parameter x is introduced. The two point function (1.111) takes the form

$$\langle T_{zz}(q)T_{zz}(-q)\rangle = \frac{D}{8}\int_0^1 dx \int d^dk\frac{(k^1-ik^2)^4 + x^2(1-x)^2\bar{q}^4}{(k^2+\Delta)^2}, \qquad (1.112)$$

with k and Δ given by (1.79) and (1.81) respectively. The terms with odd powers of k are zero because of symmetry, while the quadratic terms cancel among thems, as in (1.80) . It is not difficult to prove that the quadratic term in k vanishes using

$$\int d^dk\frac{k^ak^bk^ck^d}{(k^2+\Delta)^2} = \frac{\pi^{\frac{d}{2}}}{4}\Gamma\left(-\frac{d}{2}\right)\Delta^{\frac{d}{2}}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \qquad (1.113)$$

Using (1.83), one finds

$$\langle T_{zz}(q)T_{zz}(-q)\rangle_\delta = \frac{\pi}{48}D\frac{\bar{q}^3}{q}. \qquad (1.114)$$

Using twice the energia-momentum tensor conservatoin $qT_{zz}(q)+\bar{q}T_{z\bar{z}}(q) = 0$, it can be found

$$\langle T_{z\bar{z}}(q)T_{z\bar{z}}(-q)\rangle_\delta = \frac{\pi}{48}Dq\bar{q}. \qquad (1.115)$$

Writting this equation in coordinate space

$$\langle T_{z\bar{z}}(z)T_{z\bar{z}}(0)\rangle_\delta = -\frac{\pi D}{12}\sqrt{|g|}\Delta\delta^{(2)}(z), \qquad (1.116)$$

where it was used

$$\Delta\delta^2(\sigma) = -\frac{1}{\sqrt{|g|}}\int \frac{d^2q}{(2\pi)^2}\frac{q\bar{q}}{4}e^{iq\cdot z}. \qquad (1.117)$$

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From (1.108), an expression for $\langle T_{z\bar{z}} \rangle$ can be found

$$\langle T_{z\bar{z}} \rangle_{e^{2\omega}\delta} = \frac{D}{12}\sqrt{|g|}\Delta\omega. \tag{1.118}$$

Using the following expression for the two-dimensional scalar curvature

$$R^{(2)} = -2\Delta\omega, \tag{1.119}$$

which is valid in conformal gauge, one can write
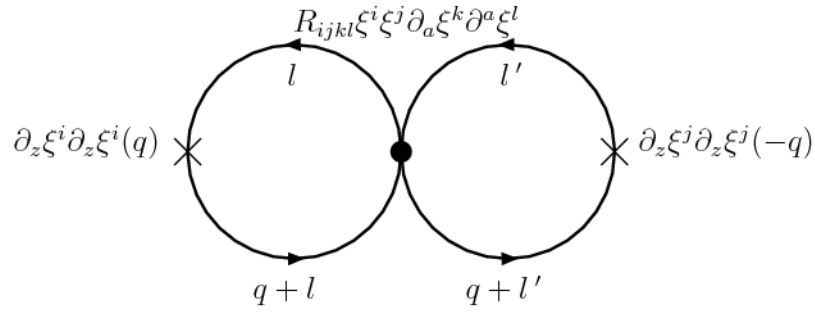
$$\langle T_{z\bar{z}} \rangle_{e^{2\omega}\delta} = -\frac{D}{24}\sqrt{|g|}R^{(2)}. \tag{1.120}$$

Multiplying both sides of (1.120) by $2g^{z\bar{z}}$
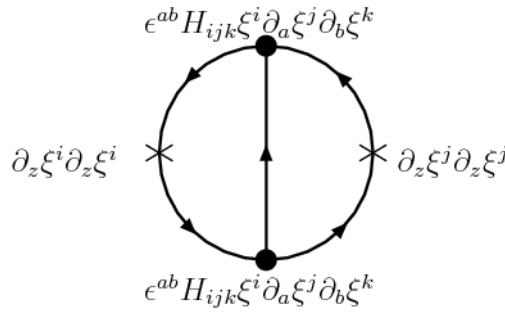
$$\langle T_a{}^a \rangle = -\frac{D}{12}R^{(2)}. \tag{1.121}$$

This expression is modified as $D \to D - 26$ by considering the ghost fields that appear when fixing the conformal gauge.

This last contribution to the trace of the energy-momentum tensor as a different structure, since it is proportional to $R^{(2)}$ and neither to $\partial_a X_0^m \partial^a X_0^n$ nor $\epsilon^{ab}\partial_a X_0^m \partial_b X_0^n$. Moreover, in this contribution do not appear terms containing G neither H. To find their contributions proportional to $R^{(2)}$, it is necessary to go to higher order terms in the expansions of $S_G[X]$ and $S_B[X]$. Those contributions can be found computing the graphs [6]

$$\tag{1.122}$$

$$\tag{1.123}$$

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$$\epsilon^{ab}H_{ijk}\xi^i\partial_a\xi^j\partial_b\xi^k \qquad\qquad \epsilon^{ab}H_{ijk}\xi^i\partial_a\xi^j\partial_b\xi^k$$

$$\partial_z\xi^i\partial_z\xi^i \qquad\qquad \partial_z\xi^j\partial_z\xi^j$$

$$\tag{1.124}$$

These two loops computations are more complicated, since subdivergences can appear. Nevertheless, according with [6] their contributions to the trace $\langle T_a{}^a\rangle$ can be computed using the same strategy of the conservation of the energy-momentum tensor. They contribute

$$\langle T_a{}^a\rangle = \alpha'\left(R - \frac{H^2}{12}\right)R^{(2)}, \tag{1.125}$$

where R is the scalar curvature of space-time.

There are remaning contributions that can appear considering the two-point functions $\langle T_{z\bar z}^\Phi(z)T_{z\bar z}^\Phi(z')\rangle$ and $\langle T_{z\bar z}^G(z)T_{z\bar z}^\Phi(z')\rangle$. From the expansion for the action with Φ is not difficult to find

$$
\begin{aligned}
T_{z\bar z}^\Phi &= \frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{z\bar z}}\left(-\frac{1}{2\pi}\int d^2z\sqrt{g}R^{(2)}\nabla_i\Phi\xi^i\right)\\
&= 2[\partial\bar\partial(\nabla_i\Phi)\xi^i + \partial(\nabla_i\Phi)\bar\partial\xi^i + \bar\partial(\nabla_i\Phi)\partial\xi^i + (\nabla_i\Phi)\partial\bar\partial\xi^i].
\end{aligned}
\tag{1.126}
$$

Taking just the last term in the last equation

$$\langle T_{z\bar z}^\Phi(q)T_{z\bar z}^\Phi(-q)\rangle_\delta = \frac{1}{4}\nabla_i\Phi\nabla_j\Phi\langle\Delta\xi^i\Delta\xi^j\rangle_\delta \tag{1.127}$$

which can be represented by the following diagram

$$\nabla_i\Phi\Delta\xi^i \qquad\qquad \nabla_j\Phi\Delta\xi^j$$

$$\tag{1.128}$$

from which it is obtained
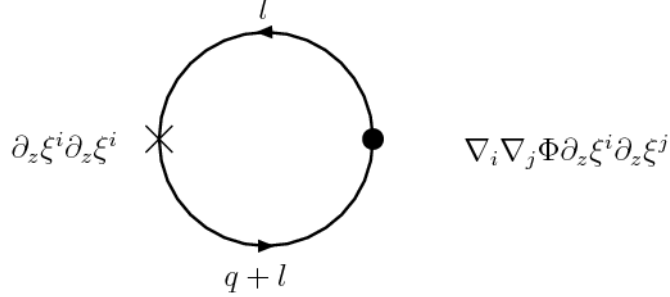
$$\langle T_{z\bar z}^\Phi T_{z\bar z}^\Phi\rangle_\delta = \pi\alpha'(\nabla\Phi)^2 q\bar q. \tag{1.129}$$

Using the energy-momentum conservation twice, a contribution of $-2\alpha'(\nabla\Phi)^2 R^{(2)}$ is obtained for the trace of the energy-momentum tensor.

Nevertheless, the expansion for the energy-momentum tensor coming from variating S_Φ also contributes at α' order, so it is necessary to compute $\langle T_{zz}^G T_{zz}^\Phi \rangle$. To compute T_{zz}^Φ the same procedure that allowed to find contributions coming from the variation of the action including Φ can be used. The following result is found

$$
\begin{aligned}
T_{zz}^\Phi &= \frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{zz}}\left(-\frac{1}{4\pi}\int d^2 z\sqrt{g}R^{(2)}\nabla_i\Phi\nabla_j\Phi\xi^i\xi^j\right) \\
&= -[\partial^2(\nabla_i\nabla_j\Phi)\xi^i\xi^j + 4\partial(\nabla_i\nabla_j\Phi)\partial\xi^i\xi^j + 2(\partial_i\partial_j\Phi)\xi^i\partial^2\xi^j \\
&\quad +2(\nabla_i\nabla_j\Phi)\partial\xi^i\partial\xi^j].
\end{aligned}
\tag{1.130}
$$

Out of these terms, only the last two will give a non-zero contribution. The product of the last term with T_{zz}^G is represented in the following diagram

$$
\tag{1.131}
$$

while the term before the last in (1.130) with T_{zz}^G can be represented by a similar diagram. From the last diagram can it can be found

$$
\langle T_{zz}^G T_{zz}^\Phi \rangle_\delta = -\frac{1}{2}\alpha'\nabla^2\Phi\int dl d\bar{l}\frac{\bar{l}^2(\bar{l}+\bar{q})^2}{l^2(l+q)^2},
\tag{1.132}
$$

this integral is the same as in (1.111). Then, using the same result one finds

$$
\langle T_{zz}^G T_{zz}^\Phi \rangle_\delta = -\frac{\pi\alpha'}{6}\nabla^2\Phi\frac{(\bar{q})^3}{q}
\tag{1.133}
$$

To compute the remaining contribution

$$
\int dl d\bar{l}\frac{\bar{l}^3(\bar{l}+\bar{q})}{l^2(l+q)^2}
\tag{1.134}
$$

has to be computed. The result is the double of (1.133) and adding up everything, one finds

$$
\langle T_{z\bar{z}} \rangle_{e^{2\omega}\delta} = 2\alpha'\sqrt{|g|}\nabla^2\Phi R^{(2)}.
\tag{1.135}
$$

Adding up all the computed contributions as indicated in (1.109) to the trace of the energy-momentum tensor it can be written as

$$\langle T_a{}^a \rangle = \beta^G_{mn} g^{ab} \partial_a X_0^m \partial_b X_0^n + \beta^B_{mn} \epsilon^{ab} \partial_a X_0^m \partial_b X_0^n + \beta^\Phi R^{(2)}, \qquad (1.136)$$

with

$$\beta^G_{mn} = R_{mn} - \frac{1}{4} H_{mlp} H_n{}^{lp} + 2\nabla_m \nabla_n \Phi, \qquad (1.137)$$

$$\beta^B_{mn} = -\frac{1}{2} \nabla^p H_{mnp} + \nabla_l \Phi H^l{}_{mn}, \qquad (1.138)$$

$$\beta^\Phi = -\frac{D-26}{12} + \alpha' \left(R - \frac{H^2}{12} + 4\nabla^2 \Phi - 4(\nabla\Phi)^2 \right). \qquad (1.139)$$

In the following, it will be shown that these beta functions can be consistently set to zero, in such a way that the theory has no conformal anomaly at one-loop.

Consistency conditions of the Weyl invariance

When the fields B_{mn} and Φ are zero and G_{mn} is the flat metric, the coefficients β^G_{mn}, β^B_{mn} in (1.136) are zero, and β^Φ reduces to a number proportional to $D-26$. Nevertheless when the bosonic string is coupled to space-time fields, there appear terms in the trace of the energy-momentum tensor which do not have this property: they are proportional to $\partial_a X^m \partial_b X^n$. Furthermore, β^Φ appears as corrections to the $D-26$ term. In order to have an anomaly free theory, all of the three β functions must cancel, in a consistent way, where by consistent is meant to preserve the property of β^Φ being a number. The term of order α' in β^Φ includes fields in spacetime, then in principle would not be constant numbers. Nevertheless, as will be shown in the following the conditions $\beta^G_{mn} = 0$ and $\beta^B_{mn} = 0$ imply that the gradient of β^Φ is zero, therefore β^Φ is a constant.

Using the Bianchi identity for the curvature tensor

$$\nabla_{[l} R_{mn]p\eta} = 0, \qquad (1.140)$$

is easy to see that the Ricci R_{mn} tensor satisfy

$$\nabla^n R_{mn} = \frac{1}{2} \nabla_n R. \qquad (1.141)$$

From the definition of H_{mnl} can be verified that it satisfies the identity

$$\nabla_{[p} H_{mnl]} = 0, \qquad (1.142)$$

what allows to write

$$\nabla^n (H_{mlp} H_n{}^{lp}) = H_{mlp} \nabla^n H_n{}^{lp} + \frac{1}{6} \nabla_m H^2, \qquad (1.143)$$

Then, computing $\nabla^n \beta^G_{mn}$ is obtained

$$\nabla^n \beta^G_{mn} = \frac{1}{2}\nabla_m R - \frac{1}{24}\nabla_m H^2 - \frac{1}{4}H_m{}^{lp}\nabla^n H_{nlp} + 2\nabla_m\nabla^2\Phi + 2R_{mn}\nabla^n\Phi, \quad (1.144)$$

where it was used the definition of the curvature tensor $[\nabla_m, \nabla_n]v^l = -R_{mnp}{}^l v^p$. In terms of the beta functions β^G_{mn} and β^B_{mn} the equation (1.144) is written as

$$\nabla^n \beta^G_{mn} = \frac{1}{2}\nabla_m\left(R - \frac{1}{12}H^2 + 4\nabla^2\Phi - 4(\nabla\Phi)^2\right) + \frac{1}{2}\beta^B_{nl}H_m{}^{nl} + 2\beta^G_{mn}\nabla^n\Phi, \quad (1.145)$$

or

$$\nabla^n \beta^G_{mn} = \frac{1}{2\alpha'}\nabla_m\beta^\Phi + \frac{1}{2}\beta^B_{nl}H_m{}^{nl} + 2\beta^G_{mn}\nabla^n\Phi, \quad (1.146)$$

which implies β^Φ is a constant if β^G_{mn} and β^B_{mn} are zero. Therefore, when all the β functions are zero the theory has Weyl symmetry at the quantum level.

Spacetime Effective Action

The system of equations obtained by the vanishing of the β functions can be written in a more suggestive way. From $\beta^\Phi = 0$ it can be found

$$R = \frac{1}{12}H^2 - 4\nabla^2\Phi - 4(\nabla\Phi)^2, \quad (1.147)$$

with this and from $\beta^G_{mn} = 0$ one obtains

$$R_{mn} - \frac{1}{2}G_{mn}R = \Theta_{mn}, \quad (1.148)$$

where

$$\Theta_{mn} \equiv \frac{1}{4}\left(H^2_{mn} - \frac{1}{6}G_{mn}H^2\right) - 2\nabla_m\nabla_n\Phi + 2G_{mn}\nabla^2\Phi - 2G_{mn}(\nabla\Phi)^2. \quad (1.149)$$

The equation (1.148), satisfied by the metric field, is the Einstein equation in space-time, with energy-momentum tensor (1.149). This is a symmetric tensor, but its conservation must be checked. In fact, applying the operator ∇^m to the equation (1.148), the left hand side is identically zero, as can be checked by using the Bianchi identity(1.140). The right hand side can be written as

$$\nabla^m\Theta_{mn} = \beta^B_{ml}H_n{}^{ml} - 2\beta^G_{mn}\nabla^m\Phi. \quad (1.150)$$

So the conditions for preserving the Weyl symmetry at the quantum level $\beta_{mn}^G = 0$, $\beta_{mn}^B = 0$ guarantee the conservation for the matter energy-momentum tensor.

Now is possible to find another two equations for the other fields in spacetime. By taking the trace of β_{mn}^G

$$G^{mn}\beta_{mn}^G = R - \frac{1}{4}H_{mn}^2 + 2\nabla^2\Phi, \tag{1.151}$$

but the condition $\beta_{mn}^G = 0$ together with $\beta^\Phi = 0$, allows to eliminate R to write

$$\nabla^2\Phi - 2(\nabla\Phi)^2 = -\frac{1}{12}H^2. \tag{1.152}$$

what constitutes the equation of motion for the dilaton. Finally, the equation for the field H will come from the condition $\beta_{mn}^B = 0$:

$$\nabla^l H_{lmn} = 2\nabla^l \Phi H_{lmn}. \tag{1.153}$$

As is known, Einstein's equations can be derived as an equation of motion of the Einstein-Hilbert action. It is tempting to think that the three equations for the spacetime fields (1.148), (1.152) and (1.153) could be deduced from an action principle. This work was done by Metsaev and Tseytlin [7], where the following action was found

$$S = \int d^{26}X\sqrt{|G|}e^{-2\Phi}\left(R - \frac{1}{12}H^2 + 4(\nabla\Phi)^2\right). \tag{1.154}$$

By making the variation of (1.154) with respect of the fields G H and Φ:

$$\begin{aligned}
\delta S &= \int d^D X\sqrt{|G|}e^{-2\Phi}\{\delta G^{mn}[R_{mn} - \frac{1}{2}RG_{mn} + 2\nabla_m\nabla_n\Phi - 2G_{mn}\nabla^2\Phi \\
&\quad +2G_{mn}(\nabla\Phi)^2 + \frac{1}{24}G_{mn}H^2 - \frac{1}{4}H_{mn}^2] - \frac{1}{2}\delta\Phi\beta^\Phi - \\
&\quad \frac{1}{6}H^{mnl}\delta H_{mnl}\}.
\end{aligned} \tag{1.155}$$

The sector of derivatives of the field G_{mn} in (1.154) does not have the form of the Einstein-Hibert action because of the exponential of the field Φ, but by making a transformation $G_{mn} = e^\omega \tilde{G}_{mn}$, (1.154) takes the form

$$\begin{aligned}
S &= \int d^D X\sqrt{|\tilde{G}|}e^{-2\Phi}e^{\omega\frac{D-2}{2}}\{\tilde{R} - (D-1)\widetilde{\nabla^2\omega} - \frac{1}{4}(D-1)(D-2)\widetilde{(\nabla\omega)^2} \\
&\quad +4\widetilde{(\nabla\Phi)^2} - \frac{1}{12}e^{-2\omega}\tilde{H}^2\},
\end{aligned} \tag{1.156}$$

where the notation with $\tilde{}$ indicates that the indices are contracted with the metric $\tilde{G}$. The Einstein-Hilbert action will be contained in (1.156) by choosing

$$\omega = \frac{4\Phi}{D-2}, \tag{1.157}$$

in that way (1.156) can be written as

$$S = \int d^D X \sqrt{|\tilde{G}|} \left(\tilde{R} - \frac{4}{D-2} \widetilde{(\nabla \Phi)^2} - \frac{1}{12} e^{\frac{8\Phi}{D-2}} \tilde{H}^2 \right). \tag{1.158}$$

In the action (1.158) can be identified a kinetic term for the Dilaton and a Maxwell term for the antisymmetric field, but with a coupling to the dilaton.

1.1.2 Superstring Sigma model

Phenomenologically, superstrings are more interesting than bosonic strings since they contain fermions in their spectrum. Before concentrating on the superstring sigma model, it will be given a brief description of the superstrings in Ramond-Neveu-Schwarz and Green-Schwarz formalism [8], [9].

In the first half of the seventies *supersymmetry* was discovered within string theory in an attempt to construct a more realistic theory which could incorporate fermions in it's spectrum. This more elaborated version of the string incorporating fermions is known as the superstring, and differently from the bosonic string, which is defined in 26 space-time dimensions to cancel the conformal anomaly, the vanishing of the superconformal anomaly makes the superstring live in 10 space-time dimensions. The first formalism used for describing the superstring dates back to that decade and is known a the RNS formalism standing for Ramond-Neveu-Schwarz. It has $N = 1$ superconformal symmetry at the world-sheet level and is also space-time supersymmetric, although this feature is rather involved. Nevertheless, the covariant quantization in this formalism is a straightforward task.

In the eighties superstring theory gained more interest. Green and Schwarz [10] showed that the theory is free of gauge, gravitational and mixed anomalies by considering it's low energy limit, which is $N = 1$ $D = 10$ super Yang-Mills theory coupled to supergravity. Also in this decade Green and Schwarz found a new formalism for the superstring, known as the GS formalism [11], which has manifest space-time supersymmetry. It has a new fermionic local symmetry at the worldsheet level, known as *Kappa symmetry* [12], which is more involved than the superconformal symmetry of the RNS formalism. For quantizing the GS formalism, one has to use the light-cone gauge so the space-time symmetries are no longer manifest. In this decade it was known the full set of superstring theories: Type I, TypeIIA, Type IIB,

Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$, which in the nineties were related to each other using dualities [13].

Ramond-Neveu-Schwarz sigma model

For the superstring, one can similarly consider the coupling to background fields corresponding to massless states, giving further information about the equations of motion for the background fields. Nevertheless, because of the complicated supersymmetric space-time structure of the RNS formalism, only some sector of the possible couplings can be turned on, namely, the NS-NS sector [14] [15]. In the conformal gauge, the sigma model action for the Heterotic string in the RNS formalism is

$$
\begin{aligned}
S &= \frac{1}{2\pi\alpha'} \int d^2z d\theta [\frac{1}{2} D\mathbf{X}^m \overline{\partial}\mathbf{X}^n (G_{mn}(\mathbf{X}) + B_{mn}(\mathbf{X})) + D\mathbf{X}^m \overline{J}^I A_{mI}(\mathbf{X}) + \theta\bar{\lambda}^{\mathcal{A}} D\bar{\lambda}^{\mathcal{A}}] \\
&+ \frac{1}{2\pi} \int d^2z r^{(2)}\Phi,]
\end{aligned} \tag{1.159}
$$

where $D = \partial_\theta + \theta\partial_z$ is the $N = 1$ $D = 2$ supersymmetric derivative, $\mathbf{X}^m = X^m + \theta\psi^m$ and $\overline{J}^I = \frac{1}{2}\mathcal{K}^I_{\mathcal{A}\mathcal{B}}\bar{\lambda}^{\mathcal{A}}\bar{\lambda}^{\mathcal{B}}$ are the Heterotic string currents that can be written in terms of the structure constants $\mathcal{K}^I_{\mathcal{A}\mathcal{B}}$ for the gauge group $E_8 \times E_8$ and right handed fermions $\bar{\lambda}^{\mathcal{A}}$, with $\mathcal{A} = 1, \ldots, 32$. A_m is the gauge potential. Besides the beta functions already written, in the absence of the Kalb-Ramond and Dilaton superfields, the check of conformal invariance allows to find a beta function associated to the gauge field: $\beta^A_{nI} = \nabla^n F_{mnI}$. The lack of manifest super-Poincaré invariance does not allow to couple the RR sector, then using this formalism there are missing equations of motion for the background fields. Therefore, the sypersymmetrical aspects of those equations of motion are missed, turning this formalism an inappropriate language for studying supersymmetrical aspect such as dualities. The use of a manifestly space-time covariant formalism is in order.

Green-Schwarz sigma model

The GS formalism makes use of superspace in 10 dimensions. For that reason, the action written using superfields makes the supersymmetry invariance manifest. Using the GS formalism, one can write a sigma model in a manifestly Super-Poincaré invariant form. The sigma model action is given by [16]

$$
S = \frac{1}{2\pi\alpha'} \int d^2z \partial Z^M \overline{\partial} Z^N (G_{MN} + B_{MN})(Z). \tag{1.160}
$$

Here Z^M stands for the $D = 10$ $N = 1$ superspace coordinates (X^m, θ^α), with $m = 0, \ldots 9$ and $\alpha = 1, \ldots 16$. G_{MN} and B_{MN} are superfields whose content is the supergravity multiplet. Some attempts have been taken to compute the beta functions in this formalism, see [17] [18]. Nevertheless, its quantization in a manifest super-Poincaré invariant way is an unsolved problem that rises difficulties in the computation of the conformal anomaly.

Besides these two formalism, there exist a formalism suitable for studying compactifications to four dimensions, known as the hybrid formalism [19]. By using this description, the Heterotic string and type II superstring beta functions were computed by studying the $N = (2, 0)$ and $N = (2, 2)$ superconformal algebra respectively [20] [21] [22].

Since one cannot couple all the background fields corresponding to the massless superstring states and covariantly quantize in ten dimensions neither with the RNS or GS formalism, one should look for a more convenient way of describing the superstring. Fortunately, there exist a formulation in which the super-Poincaré invariance is manifest and can be quantized covariantly. This is the pure spinor formulation for the superstring [23], whose sigma model for describing the Heterotic and type II superstrings [24] has been used to compute the equations of motion for the backgrounds, giving respectively the super Yang-Mills/supergravity equations of motion for the heterotic case [25] and supergravity equations of motion for the type II sigma model [26]. It is worth to note that before pure spinors were used to describe superstrings, integrability along pure spinor lines allowed to find the super Yang-Mills and supergravity equations of motion in ten dimensions [28] . Before discussing the pure spinor sigma model, which will be done in chapter two and three, it will be useful to give a brief review of the pure spinor formalism in a flat background. For detailed and pedagogical reviews, see [29] and [30].

1.2   Pure spinor formalism

The Pure Spinor formalism has its roots in the Siegel approach for describing the superstring [31]. This approach had success for covariantly quantizing the superparticle, but it could not be used to describe the physical superstring spectrum. Nevertheless it had the advantage that all the worldsheet fields are free, making trivial the computation of the OPE's. Instead of describing the Siegel approach, the pure spinor description for the Heterotic and type II superstrings will be given directly.

1.2.1 Heterotic superstring in the Pure Spinor formalism

The action for the heterotic superstring in the pure spinor formalism is given by

$$S = \frac{1}{2\pi\alpha'}\int d^2z(\frac{1}{2}\partial X^m\overline{\partial}X_m + p_\alpha\overline{\partial}\theta^\alpha + \overline{b}\partial\overline{c}) + S_\lambda + S_{\overline{J}}, \qquad (1.161)$$

where the worldsheet variables $(X^m, \theta^\alpha, p_\alpha)$, with $m = 0\ldots9$, $\alpha = 1\ldots16$, describe the $N = 1$ $D = 10$ superspace. p_α is the conjugate momentum to θ^α. This formalism takes its name from the bosonic spinor λ^α, which is constrained to satisfy the pure spinor condition $\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta = 0$, where γ^m are 16×16 symmetric ten-dimensional gamma matrices. The pure spinor part of the action, denoted by S_λ, is the action for a free β γ system, where the conjugate momentum to λ^α is denoted by ω_α. $S_{\overline{J}}$ denotes the action for the heterotic right-moving currents and $(\overline{b}, \overline{c})$ are the right moving Virasoro ghosts. It is worth to note that the Lorentz currents $N^{ab} = \frac{1}{2}\lambda\gamma^{ab}\omega$ and ghost number current $J = \lambda^\alpha\omega_\alpha$ satisfy

$$N^{mn}(y)N^{pq}(z) \rightarrow \alpha'\frac{\eta^{p[n}N^{m]q}(z) - \eta^{q[n}N^{m]p}(z)}{y - z} - 3\alpha'^2\frac{\eta^{m[q}\eta^{p]n}}{(y - z)^2}, \qquad (1.162)$$

$$J(y)J(z) \rightarrow -\frac{4\alpha'^2}{(y - z)^2}. \qquad (1.163)$$

It is worth to note that the -3 coefficient in the double pole, added with a $+4$ coefficient for the double pole of the Lorentz curent in Siegel approach $M^{mn} = \frac{1}{2}p\gamma^{mn}\theta$, gives a $+1$ coefficient, which is the same as in the Lorentz current of the RNS formalism. These currents have OPEs with the pure spinors

$$N^{mn}(y)\lambda^\alpha(z) \rightarrow \frac{1}{2}\alpha'(\gamma^{mn})^\alpha{}_\beta\frac{\lambda^\beta(z)}{y - z}, \quad J(y)\lambda^\alpha(z) \rightarrow \alpha'\frac{\lambda^\alpha(z)}{y - z}, \qquad (1.164)$$

while the right-moving currents satisfy

$$\overline{J}^I(y)\overline{J}^J(z) \rightarrow \alpha'\frac{f^{IJ}{}_K\overline{J}^K(z)}{\overline{y} - \overline{z}} + \alpha'^2\frac{\delta^{IJ}}{(\overline{y} - \overline{z})^2}, \qquad (1.165)$$

where $f^{IJ}{}_K$ are the $E_8 \times E_8$ structure constants. Physical states are defined as vertex operators in the cohomology of the BRST charge* $Q = \oint dz\lambda^\alpha d_\alpha$ and $\overline{Q} = \oint(\overline{c}\overline{T} + \overline{c}\overline{\partial}\overline{c}\overline{b})$, where d_α are the worldsheet variables corresponding to $N = 1$ $D = 10$ space-time supersymmetric derivatives and is given by

$$d_\alpha = p_\alpha - \frac{i}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m + \frac{1}{8}\gamma^m_{\alpha\beta}(\gamma_m)_{\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta. \qquad (1.166)$$

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*For a reference of BRST quantization of the superstring and a proof of equivalence of the pure spinor formalism and GS formalism, see [32] and [33].

1.2.2 Type II superstring in the Pure Spinor formalism

The pure spinor closed string action in flat space-time is defined by using the super-space coordinates X^m with $m = 0, \ldots, 9$ and the conjugate pairs $(p_\alpha, \theta^\alpha), (\widetilde{p}_{\overline{\alpha}}, \widetilde{\theta^{\overline{\alpha}}})$ with $(\alpha, \overline{\alpha}) = 1, \ldots, 16$. For the type IIA superstring the spinor indices α and $\overline{\alpha}$ have the opposite chirality while for the type IIB superstring they have the same chirality. In order to define a conformal invariant system we need to include a pair of pure spinor ghost variables $(\lambda^\alpha, \omega_\alpha)$ and $(\widetilde{\lambda^{\overline{\alpha}}}, \widetilde{\omega}_{\overline{\alpha}})$. These ghosts are constrained to satisfy the pure spinor conditions $(\lambda \gamma^m \lambda) = (\widetilde{\lambda} \gamma^m \widetilde{\lambda}) = 0$, where $\gamma^m_{\alpha\beta}$ and $\gamma^m_{\overline{\alpha}\overline{\beta}}$ are the 16×16 symmetric ten dimensional gamma matrices. Because of the pure spinor conditions, ω and $\widetilde{\omega}$ are defined up to $\delta\omega = (\lambda \gamma^m)\Lambda_m$ and $\delta\widetilde{\omega} = (\widetilde{\lambda}\gamma^m)\widetilde{\Lambda}_m$. The quantization of the model is performed after the construction of the BRST-like charges $Q = \oint \lambda^\alpha d_\alpha, \widetilde{Q} = \oint \widetilde{\lambda^{\overline{\alpha}}} \widetilde{d}_{\overline{\alpha}}$, here d_α and $\widetilde{d}_{\overline{\alpha}}$ are the world-sheet variables corresponding to the $N = 2$ $D = 10$ space-time supersymmetric derivatives and are supersymmetric combinations of the space-time superspace coordinates of conformal weights $(1,0)$ and $(0,1)$ respectively. The action in flat space is a free action involving the above fields, that is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, (\frac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial} \theta^\alpha + \widetilde{p}_{\overline{\alpha}} \partial \widetilde{\theta^{\overline{\alpha}}}) + S_{pure}, \qquad (1.167)$$

where S_{pure} is the action for the pure spinor ghosts.
The left $N^{mn} = \frac{1}{2}\lambda \gamma^{mn} \omega$ and right-moving currents $\widetilde{N}^{mn} = \frac{1}{2}\widetilde{\lambda}\gamma^{mn}\widetilde{\omega}$ satisfy the OPE's

$$N^{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}\alpha'(\gamma^{mn})^\alpha{}_\beta \frac{\lambda^\beta(z)}{y-z}, \quad \widetilde{N}^{mn}(y)\widetilde{\lambda^{\overline{\alpha}}}(z) \to \frac{1}{2}\alpha'(\gamma^{mn})^{\overline{\alpha}}{}_{\overline{\beta}} \frac{\widetilde{\lambda^{\overline{\beta}}}(z)}{\overline{y}-\overline{z}} \quad (1.168)$$

$$N^{mn}(y)N^{pq}(z) \to \alpha' \frac{\eta^{p[n}N^{m]q}(z) - \eta^{q[n}N^{m]p}(z)}{y-z} - 3\alpha'^2 \frac{\eta^{m[q}\eta^{p]n}}{(y-z)^2}, \qquad (1.169)$$

$$\widetilde{N}^{mn}(y)\widetilde{N}^{pq}(z) \to \alpha' \frac{\eta^{p[n}\widetilde{N}^{m]q}(z) - \eta^{q[n}\widetilde{N}^{m]p}(z)}{\overline{y}-\overline{z}} - 3\alpha'^2 \frac{\eta^{m[q}\eta^{p]n}}{(\overline{y}-\overline{z})^2}, \qquad (1.170)$$

Having a covariantly quantized description for the superstring brings important advantages. Scattering amplitudes have been computed up to two loops, [34], [35], [36], [37], [38], [39] and [40]. Also it has been possible to study the superstring in a curved background more properly, that means, including the full $D = 10$ $N = 1$ supermultiplet and finding their equations of motion. In this area, is of importance to know the effective field theories for the massless modes of the string. One reason is to know the genuine effective stringy effects in the theory. A second reason is that

it would be possible to test duality conjectures beyond the leading order and also would be interesting to know the effects of the string corrections on the solutions of the supergravity equations of motion.

As emphasized, these Ph.D thesis relays on the area of the non-linear sigma model for the superstring in the pure spinor description. The coupling of the pure spinor superstring to a generic background, including RR fields, was given for the first time by Berkovits and Howe [24], where also a set of $D = 10$ $N = 1$ super-Yang-Mills and supergravity constraints were computed in the heterotic superstring case by studying the nilpotency of the pure spinor BRST charge and the conservation of its respective BRST current. Also they found a set $D = 10$ $N = 2$ supergravity constraints analog considerations for the type II superstring in a generic background. Both the heterotic and type II pure spinor superstrings in a generic background will be reviewed in chapter 2 and 3. For the open superstring in the pure spinor description coupled to a background, it was shown that the classical BRST invariance implies that the background fields satisfy the full non-linear supersymmetric Born-Infeld equations of motion [41]. The one-loop beta functions for the heterotic superstring using the pure spinor formalism were computed by Chandia and Vallilo [25]. These authors also show that the sYM/supergravity constraints makes the beta functions to be zero, implying in conformal invariance at one-loop. In collaboration with Chandia [26] the one-loop beta functions for the type II superstring were computed, and also were verified the conformal invariance of this theory by using the lowest order $D = 10$ $N = 2$ supergravity constraints. This will be developed in detail in chapter 3 of this thesis. It is worth to note that using the pure spinor formalism, the full superfield multiplet can be coupled to the superstring. So this allowed to compute covariantly the equations of motion for the background superfields, even the RR superfields in a manifestly covariant manner[†].

There is one more study that can be performed using the superstring sigma model using the pure spinor formalism. The Green-Schwarz mechanism demands an anomalous transformation of the the Kalb-Ramond superfield [10], which amounts to an α' order Chern-Smons modifications of the field-strength related to this superfield. It will now be explained how to compute such α' corrections. Using the pure spinor sigma model it was shown at the lowest order in α', that the BRST invariance puts the background fields on-shell [24]. It is in the quantum regime of the BRST invariance that it is expected to find the Chern-Simons modifications. These Chern-

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[†]For further studies of the pure spinor superstring in a generic background see [27] and references therein.

Simons modifications are of two types: Yang-Mills and Lorentz, as is known since the Green-Schwarz mechanism for the cancellation of gauge, gravitational and mixed anomalies in the framework of the ten-dimensional low-energy effective field theories [10]. Also Hull and Witten [42] noted the appearance of those modifications to cancel the sigma model gauge anomalies. Atick, Dhar and Ratra [43] gave further evidence for the existence of the Yang-Mills Chern-Simons modification by making a Superspace description of $\mathcal{N} = 1$ supergravity coupled to $\mathcal{N} = 1$ super Yang-Mills. The Chern-Simons modifications were even noted in the component formulation of supergravity in order to have a consistent theory [44]. Furthermore, integrability along pure spinor lines allowed Howe [45] to incorporate the Chern-Simons corrections. By studying what conditions are imposed on the background superfields by the preservation of the BRST invariance properties at first order in α' for the Heterotic sigma model in the pure spinor description, the Yang-Mills Chern-Simons modifications have been computed in as-yet unpublished work [46] which will be described in detail in chapter 4. The Lorentz Chern-Simons modifications constitute work in progress and will be discussed in chapter 5. Both Yang-Mills and Lorentz Chern-Simons modifications appear as stringy corrections to some of the classical SYM/SUGRA constraints mentioned in the preceding paragraph. Besides the Chern-Simons modifications, other corrections are expected to preserve supersymmetry. Being a manifestly supersymmetric, it seems promising that the pure spinor sigma model would be useful to find a complete α' correction preserving supersymmetry in space-time. This will help to settle an old debate found in the literature, discussed in some works of Gates et al. [47] , [48] and [49] on one hand and Bonora et al. [50], [51], [52], [53] and [54] where two sets of string corrected constraints cannot be related among them.

Recently a new set of supergravity constraints have been introduced by Lechner and Tonin [55] and it will be of interest to compare their α' corrections with those computed directly from the pure spinor superstring.

Chapter 2

Ten-Dimensional Supergravity Constraints from the Pure Spinor Formalism for the Heterotic Superstring

Before discussing the conformal anomaly and the gauge and Lorentz anomaly for the superstring, it will be introduced the sigma-model type action in the pure spinor formalism [24] for the heterotic superstring. As in the bosonic string case [56], the starting point for costructing a sigma-model action are the integrated vertex operators corresponding to the massless states. This chapter is fully based on [24] and the pourpose of including it in the thesis is to make the text more complete and set notation, instead of being original in this topic.

2.1   Vertex Operators in the pure spinor formalism

The massless supergravity and super Yang-Mills vertex operators are respectively given by

$$V_{SG} = \int d^2z[\partial\theta^\alpha A_{\alpha m}(x,\theta) + \Pi^n A_{nm}(x,\theta) + d_\alpha E_m^\alpha(x\theta) + N_{np}\Omega_m{}^{np}]\overline{\partial}x^m, \quad (2.1)$$

$$V_{sYM} = \int d^2z[\partial\theta^\alpha A_{\alpha I}(x,\theta) + \Pi^n A_{nI}(x,\theta) + d_\alpha W_I^\alpha(x\theta) + N_{np}U_I{}^{np}]\overline{J}^I, \quad (2.2)$$

where the last two terms in each vertex operators are present to make them BRST invariant and $\Pi^m = \partial x^m + \frac{i}{2}\theta\gamma^m\partial\theta$. Note that any of this two vertex operators could be constructed from the open string vertex operator

$$V_{open} = \int dz[\partial\theta^\alpha A_\alpha(x,\theta) + \Pi^n A_n(x,\theta) + d_\alpha W^\alpha(x\theta) + N_{np}U^{np}] \quad (2.3)$$

multiplying either with $\int d\bar{z}\overline{\partial}x^m$ or $\int d\bar{z}\overline{J}$. By computing the conditions for BRST invariance of (2.1) one finds

$$(\gamma_{npqrs})^{\alpha\beta}D_\alpha A_{\beta m} = 0, \qquad \partial^m(\partial_m A_{\beta n} - \partial_n A_{\beta m}) = 0, \tag{2.4}$$

which are the linearized $N = 1$ supergravity equations motions, and also

$$A_{nm} = -\frac{i}{8}D_\alpha(\gamma_n)^{\alpha\beta}A_{\beta m}, \;\; E_m{}^\beta = -\frac{i}{10}(\gamma^n)^{\alpha\beta}(D_\alpha A_{nm} - \partial_n A_{\alpha m}), \tag{2.5}$$

$$\Omega_m{}^{np} = \frac{1}{8}D_\alpha(\gamma^{np})^\alpha{}_\beta E_m{}^\beta = \partial_{[n}A_{p]m},$$

which defines the linearized supergravity connections and field-strengths in terms of $A_{\alpha m}$. Similarly, the BRST invariance of (2.2) leads to the linearized $N = 1$ super Yang-Mills equation of motion

$$(\gamma_{npqrs})^{\alpha\beta}D_\alpha A_{\beta I} = 0, \tag{2.6}$$

as can be read-off from the condition $\lambda^\alpha\lambda^\beta D_\alpha A_{\beta I} = 0$ using the pure spinor condition; and also to the definitions of the linearized super Yang-Mills connections and field strengths in terms of $A_{\alpha I}$:

$$A_{nI} = -\frac{i}{8}D_\alpha(\gamma_n)^{\alpha\beta}A_{\beta I}, \;\; W_I{}^\beta = -\frac{i}{10}(\gamma^n)^{\alpha\beta}(D_\alpha A_{nI} - \partial_n A_{\alpha I}), \tag{2.7}$$

$$U_I{}^{np} = \frac{1}{8}D_\alpha(\gamma^{np})^\alpha{}_\beta W_I^\beta = \partial_{[n}A_{p]I}.$$

The on-shell graviton h_{mn} is contained in the $(\gamma^n\theta)_\alpha h_{mn}(x)$ of $A_{\alpha m}(x,\theta)$, while the on-shell gluon a_{nI} is in the $(\gamma^n\theta)_\alpha a_{nI}(x)$ of $A_{\alpha I}(x,\theta)$.

By considering the coupling of the superstring to a generic background, (2.4)-(2.7) will be generalized to covariant non-linear equations.

2.2 Heterotic Superstring in a Generic Background

By adding the supergravity and super Yang-Mills vertex operators (2.1) and (2.2) to the flat action (1.161) and covariantizing respect to $N = 1$ $D = 10$ super-reparameterization invariance, one can arrive to an action for the coupling of the Heterotic superstring to a curved background. Also, one can consider the worldsheet fields ∂x^m, $\overline{\partial}x^m$, $\partial\theta^\alpha$, $\overline{\partial}\theta^\alpha$, d_α, $\overline{J}$ and $\lambda^\alpha\omega_\beta$. Then, by making products among them, one can write an expression which is classically invariant under worldsheet conformal transformations. The action is given by

$$S = \frac{1}{2\pi\alpha'}\int d^2z[\frac{1}{2}\partial Z^M\overline{\partial}Z^N(G_{NM} + B_{NM}) + d_\alpha\overline{\partial}Z^M E_M{}^\alpha + \partial Z^M\overline{J}^I A_{MI} \tag{2.8}$$

$$+ d_\alpha\overline{J}^I W_I^\alpha + \lambda^\alpha\omega_\beta\overline{J}^I U_{I\alpha}{}^\beta + \lambda^\alpha\omega_\beta\overline{\partial}Z^M\Omega_{M\alpha}{}^\beta] + S_{FT} + S_{ghost} + S_\lambda + S_{\overline{J}}.$$

In this notation $Z^M = (x^m, \theta^\mu)$ are coordinates for the superspace. Middle alphabet indices denote the curved superspace indices, while beginning alphabet indices $a = (a, \alpha)$ denote the tangent superspace indices. The set of background superfields is given by G_{MN}, B_{MN}, $E_M{}^\alpha$, A_{MI}, W_I^α, $U_{I\alpha}{}^\beta$, $\Omega_{M\alpha}{}^\beta$ and Φ. In terms of the supervielbein $E_M{}^A$, G_{MN} is given by $G_{MN} = E_M{}^a E_N{}^b \eta_{ab}$. B_{MN} is the two-form Kalb-Ramond potential, and Φ is the dilaton. A_{MI} is the super Yang-Mills potential, while W_I^α and $U_{I\alpha}{}^\beta$ will be related to the super Yang-Mills field strengths. S_{FT} denotes the Fradkin-Tseytlin action $S_{FT} = \frac{1}{4\pi} \int d^2 z \Phi(z) r^{(2)}$, where $r^{(2)}$ is the two-dimensional scalar curvature. Finally, $\Omega_{M\alpha}{}^\beta$ is the sping connection superfield. Because of the form that ω_α appears in (2.8) and the pure spinor condition, there is a gauge invariance $\delta\omega_\alpha = \Lambda^a(\gamma_a\lambda)_\alpha$, so the background superfields satisfy $(\gamma^{bcde})_\beta{}^\alpha \Omega_{M\alpha}{}^\beta = (\gamma^{bcde})_\beta{}^\alpha U_{I\alpha}{}^\beta = 0$, which imply

$$\Omega_{M\alpha}{}^\beta = \Omega_M^{(s)}\delta_\alpha{}^\beta + \frac{1}{4}\Omega_M{}^{cd}(\gamma_{cd})_\alpha{}^\beta, \quad U_{I\alpha}{}^\beta = U_I^{(s)}\delta_\alpha{}^\beta + \frac{1}{4}U_I{}^{cd}(\gamma_{cd})_\alpha{}^\beta. \tag{2.9}$$

The action (2.8) is invariant under local gauge transformations

$$\delta E_M{}^b = \eta_{cd}\Lambda^{bc}E_M{}^d, \quad \delta E_M{}^\alpha = \Sigma_\beta^\alpha E_M{}^\beta, \quad \delta\Omega_{M\alpha}{}^\beta = \partial_M\Sigma_\alpha{}^\beta + \Sigma_\alpha^\gamma\Omega_{M\gamma}{}^\beta - \Sigma_\gamma^\beta\Omega_{M\alpha}{}^\gamma,$$

$$\delta W_I^\alpha = \Sigma_\gamma^\alpha W_I^\gamma, \quad \delta U_{I\alpha}{}^\beta = \Sigma_\alpha^\gamma U_{I\gamma}{}^\beta - \Sigma_\gamma^\beta U_{I\alpha}{}^\gamma, \quad \delta\lambda^\alpha = \Sigma_\gamma^\alpha\lambda^\gamma, \quad \delta\omega_\alpha = -\Sigma_\alpha^\gamma\omega_\gamma, \tag{2.10}$$

as well as under local shift transformations.

$$\delta\Omega_\alpha^{(s)} = 4(\gamma_c)_{\alpha\beta}h^{c\beta}, \quad \delta\Omega_\alpha{}^{bc} = 2(\gamma^{[b})_{\alpha\beta}h^{c]\beta}, \quad \delta d_\alpha = -\delta\Omega_{\alpha\beta}{}^\gamma\lambda^\beta\omega_\gamma, \quad \delta U_{I\alpha}{}^\beta = W_I^\gamma\delta\Omega_{\gamma\alpha}{}^\beta, \tag{2.11}$$

where the transformation of $\Omega_{\alpha\beta}{}^\gamma$ has been chosen in such a way for not to change the pure spinor BRST current.

2.2.1 Heterotic Nilpotency Constraints

The constraints found by requiring that the BRST charge remains nilpotent when the string is coupled to a curved background can be found either by using canonical commutation relations [24], computing directly twice the BRST variation on various worldsheet fields in (2.8) [57] or by a tree level computation, as explained in chapter 4. In this section we use the first approach, with a commutator algebra

$$[P_M, Z^N] = \delta_M{}^N, \quad [\omega_\alpha, \lambda^\beta] = \delta_\alpha^\beta, \quad [\overline{J}^I, \overline{J}^J] = f^{IJ}{}_K \overline{J}^K, \tag{2.12}$$

where the canonical momentum is defined as usual $P_M = \delta L/\delta(\partial_0 Z^M)$. By computing this momentum one finds

$$d_\alpha = E_\alpha{}^M (P_M + \frac{1}{2}(\partial Z^N - \overline{\partial} Z^N) B_{NM} - \lambda^\delta \omega_\beta \Omega_{M\delta}{}^\beta - \overline{J}^I A_{MI}). \qquad (2.13)$$

Then, one can use the commutators algebra (2.12) to find

$$\{Q, Q\} = \oint \lambda^\alpha \lambda^\beta [T_{\alpha\beta}{}^C D_C - \frac{1}{2}(\partial Z^N - \overline{\partial} Z^N) H_{\alpha\beta N} - \lambda^\gamma \omega_\delta R_{\alpha\beta\gamma}{}^\delta - \overline{J}^I F_{\alpha\beta I}], \quad (2.14)$$

where $D_C = E_C{}^M (P_M - \lambda^\alpha \omega_\beta \Omega_{M\alpha}{}^\beta - \overline{J}^I A_{MI})$.

From (2.14) one can read the nilpotency constraints

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^C = \lambda^\alpha \lambda^\beta H_{\alpha\beta C} = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = \lambda^\alpha \lambda^\beta F_{\alpha\beta I} = 0. \qquad (2.15)$$

2.2.2 Heterotic Holomorphicity Constraints

In this subsection it will be computed the conditions for $\overline{\partial}(\lambda^\alpha d_\alpha) = 0$ at the lowest order. Again there are three possible ways to compute this constraints. One is by using the classical equations of motion derived for the worldsheet fields in (2.8) [24], by computing directly the BRST variation of this action [57] or by computing tree level diagrams, as will be shown in chapter four. In this chapter it will be followed the first approach.

By variating λ^α and ω_α in (2.8) one obtains respectively

$$\overline{\partial}\omega_\alpha = -(\overline{\partial} Z^M \Omega_{M\alpha}{}^\beta + \overline{J}^I U_{I\alpha}{}^\beta)\omega_\beta, \ \ \overline{\partial}\lambda^\alpha = (\overline{\partial} Z^M \Omega_{M\beta}{}^\alpha + \overline{J}^I U_{I\beta}{}^\alpha)\lambda^\beta. \qquad (2.16)$$

The equations of motion for the right-moving Heterotic currents can be found by using bosonization. The result is

$$\partial \overline{J}^I = f^{IJ}{}_K \overline{J}^K (\partial Z^M A_{MJ} + d_\alpha W_J^\alpha + \lambda^\alpha \omega_\beta U_{J\alpha}{}^\beta). \qquad (2.17)$$

Finally, by computing the variation of (2.8) with respect of superspace coordinates Z^M one finds

$$\overline{\partial} d_\alpha = E_\alpha{}^P [(\partial_{[P} E_{M]}{}^a) E_N{}^b \eta_{ab} + \partial_{[P} E_{N]}{}^a E_M{}^b \eta_{ab} - \frac{1}{2} H_{PMN}) \partial Z^M \overline{\partial} Z^N \qquad (2.18)$$

$$+2(\partial_{[P} E_{N]}{}^\beta) d_\beta + \partial_{[P} \Omega_{N]\gamma}{}^\beta \lambda^\gamma \omega_\beta) \overline{\partial} Z^N - \Omega_{P\gamma}{}^\beta \overline{\partial}(\lambda^\gamma \omega_\beta) - A_{PI} \partial \overline{J}^I$$

$$+(2\partial_{[P} A_{M]I} \partial Z^M + \partial_P W_I^\beta d_\beta + \partial_P U_{I\gamma}{}^\beta \lambda^\gamma \omega_\beta) \overline{J}^I].$$

So, by using (2.16), (2.17) and (2.18) one finds the that derivative in the $\overline{z}$ direction of the BRST current is

$$\overline{\partial}(\lambda^\alpha d_\alpha) = \lambda^\alpha [\Pi^b \overline{\Pi}^c (T_{\alpha bc} + T_{\alpha cb} - H_{\alpha bc})] + \frac{1}{2}\Pi^\beta \overline{\Pi}^c (T_{\alpha\beta c} - H_{\alpha\beta c}) + d_\beta \overline{\Pi}^c T_{\alpha c}{}^\beta + \lambda^\beta \omega_\gamma \overline{\Pi}^c R_{\alpha c\beta}{}^\gamma]$$
$$(2.19)$$

$$+\lambda^{\alpha}[\Pi^{b}\overline{J}^{I}(F_{\alpha bI}+\frac{1}{2}W_{I}^{\beta}(T_{\alpha\beta b}-H_{\alpha\beta b}))+\Pi^{\gamma}\overline{J}^{I}(F_{\alpha\gamma I}+\frac{1}{2}W_{I}^{\beta}H_{\alpha\gamma\beta})]$$

$$+\lambda^{\alpha}[d_{\beta}\overline{J}^{I}(\nabla_{\alpha}W_{I}^{\beta}-T_{\alpha\gamma}{}^{\beta}W_{I}^{\gamma}-U_{I\alpha}{}^{\beta})+\lambda^{\gamma}\omega_{\delta}\overline{J}^{I}(\nabla_{\alpha}U_{I\gamma}{}^{\delta}-R_{\alpha\beta\gamma}{}^{\delta})],$$

where $\Pi^{A}=\partial Z^{M}E_{M}{}^{A}$, $\overline{\Pi}^{A}=\overline{\partial}Z^{M}E_{M}{}^{A}$ and $T_{ABc}\equiv T_{AB}{}^{d}\eta_{dc}$.

Since $\overline{\Pi}^{\alpha}$ is related to $\overline{J}^{I}$ through $\overline{\Pi}^{\alpha}=-\overline{J}^{I}W_{I}^{\alpha}$ by using the equation of motion for the worldsheet field d_{α} in (2.8), we arrive at the following set of constraints for holomorphicity of the BRST current at the lowest order in α'

$$T_{\alpha(bc)}=-H_{\alpha bc}=T_{\alpha\beta}{}^{c}-H_{\alpha\beta}{}^{c}=T_{c\alpha}{}^{\beta}=0,\ \lambda^{\alpha}\lambda^{\beta}R_{d\alpha\beta}{}^{\gamma}=0,\ F_{\alpha\beta I}=-\frac{1}{2}W_{I}^{\gamma}H_{\gamma\alpha\beta},$$
$$(2.20)$$
$$F_{\alpha bI}=-W_{I}^{\gamma}T_{\gamma\alpha b},\ \nabla_{\alpha}W_{I}^{\beta}-T_{\alpha\gamma}{}^{\beta}W_{I}^{\gamma}=U_{I\alpha}{}^{\beta},\ \lambda^{\alpha}\lambda^{\beta}(\nabla_{\alpha}U_{I\beta}{}^{\gamma}+R_{\alpha\gamma\beta}{}^{\delta}W_{I}^{\gamma})=0.$$

It will be explained in chapter 4 how to compute those constraints perturbatively in α'.

In the following chapter, it will be discussed the pure spinor sigma model for the type II superstring.

Chapter 3

One-loop Conformal Invariance of the Type II Pure Spinor Superstring in a Curved Background

Having gained experience with the Heterotic sigma model, the type II sigma model will be introduced and the conditions for conformal invariance will be computed. At the end of the chapter it will be shown how the classical constraints imply in the equations of motion for the background.

3.1 Classical Considerations

In a curved background, the pure spinor sigma model action for the type II superstring is obtained by adding to the flat action of (1.167) the integrated vertex operator for supergravity massless states and then covariantizing respect to ten dimensional $N = 2$ super-reparameterization invariance. The result of doing this is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \; (\frac{1}{2}\Pi^a\overline{\Pi}^b\eta_{ab} + \frac{1}{2}\Pi^A\overline{\Pi}^B B_{BA} + d_\alpha\overline{\Pi}^\alpha + \widetilde{d}_{\overline{\alpha}}\Pi^{\overline{\alpha}} + (\lambda^\alpha\omega_\beta)\overline{\Omega}_\alpha{}^\beta + (\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}})\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}}$$

$$(3.1)$$

$$+ d_\alpha\widetilde{d}_{\overline{\beta}}P^{\alpha\overline{\beta}} + (\lambda^\alpha\omega_\beta)\widetilde{d}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}} + (\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}})d_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + (\lambda^\alpha\omega_\beta)(\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}})S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}}) + S_{pure} + S_{FT},$$

where $\Pi^A = \partial Z^M E_M{}^A, \overline{\Pi}^A = \overline{\partial}Z^M E_M{}^A$ with $E_M{}^A$ the supervielbein and Z^M are the curved superspace coordinates, B_{BA} is the super two-form potential. The connections appears as $\overline{\Omega}_\alpha{}^\beta = \overline{\partial}Z^M\Omega_{M\alpha}{}^\beta = \overline{\Pi}^A\Omega_{A\alpha}{}^\beta$ and $\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} = \partial Z^M\widetilde{\Omega}_{M\overline{\alpha}}{}^{\overline{\beta}} = \Pi^A\widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}}$. They are independent since the action of (3.1) has two independent Lorentz symmetry transformations. One acts on the α-type indices and the other acts on the $\overline{\alpha}$-type indices. S_{pure} is the action for the pure spinor ghosts and is the same as in the flat space case of (1.167).

As was shown in [24], the gravitini and the dilatini fields are described by the lowest θ-components of the superfields $C_\alpha{}^{\beta\overline{\gamma}}$ and $\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$, while the Ramond-Ramond field

strengths are in the superfield $P^{\alpha\overline{\beta}}$. The dilaton is the theta independent part of the superfield Φ which defines the Fradkin-Tseytlin term

$$S_{FT} = \frac{1}{2\pi} \int d^2z \; r \; \Phi, \tag{3.2}$$

where r is the world-sheet curvature. Because of the pure spinor constraints, the superfields in 3.1 cannot be arbitrary. In fact, because of the gauge invariances $\delta\omega_\alpha = \Lambda^a(\gamma_a\lambda)_\alpha$ and $\delta\widetilde{\omega}_{\overline{\alpha}} = \overline{\Lambda}^a(\gamma_a\widetilde{\lambda})_{\overline{\alpha}}$ one can find

$$\Omega_{A\alpha}{}^\beta = \Omega_A\delta_\alpha{}^\beta + \frac{1}{4}\Omega_{Aab}(\gamma^{ab})_\alpha{}^\beta, \quad \widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{\Omega}_A\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{\Omega}_{Aab}(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}}, \tag{3.3}$$

$$C_\alpha{}^{\beta\overline{\gamma}} = C^{\overline{\gamma}}\delta_\alpha{}^\beta + \frac{1}{4}C_{ab}{}^{\overline{\gamma}}(\gamma^{ab})_\alpha{}^\beta, \quad \widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} = \widetilde{C}^\gamma\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{C}_{ab}{}^\gamma(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}}, \tag{3.4}$$

$$S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}} = S\delta_\alpha{}^\beta\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}S_{ab}(\gamma^{ab})_\alpha{}^\beta\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{S}_{ab}(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}}\delta_\alpha{}^\beta + \frac{1}{16}S_{abcd}(\gamma^{ab})_\alpha{}^\beta(\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}. \tag{3.5}$$

The action of (3.1) is BRST invariant if the background fields satisfy suitable constraints. As was shown in [24], these constraints imply that the background field satisfy the type II supergravity equations. The BRST invariance is obtained by requiring that the BRST currents $j_B = \lambda^\alpha d_\alpha$ and $\widetilde{j}_B = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{d}_{\overline{\alpha}}$ are conserved. Besides, the BRST charges $Q = \oint j_B$ and $\widetilde{Q} = \oint \widetilde{j}_B$ are nilpotent and anticommute. Let us review these properties now.

3.1.1 Nilpotency

As was shown in [24] (see also [57]), nilpotency is obtained after defining momentum variables in (3.1) and then using the canonical Poisson brackets. The only momentum variable that does not appear in (3.1) is the conjugate momentum of Z^M which is defined as $P_M = (2\pi\alpha')\delta S/\delta(\partial_0 Z^M)$ where $\partial_0 = \frac{1}{2}(\partial + \overline{\partial})$. It is not difficult to see that ω_α is the conjugate momentum to λ^α and that $\widetilde{\omega}_{\overline{\alpha}}$ is the one for $\widetilde{\lambda}^{\overline{\alpha}}$. Nilpotence of Q determines the constraints

$$\lambda^\alpha\lambda^\beta H_{\alpha\beta A} = \lambda^\alpha\lambda^\beta\lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = \lambda^\alpha\lambda^\beta\widetilde{R}_{\alpha\beta\overline{\gamma}}{}^{\overline{\delta}} = 0, \tag{3.6}$$

$$\lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^a = \lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^\gamma = \lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^{\overline{\gamma}} = 0, \tag{3.7}$$

where $H = dB$, the torsion $T_{AB}{}^\alpha$ and $R_{AB\gamma}{}^\delta$ are the torsion and the curvature constructed using $\Omega_{AB}{}^\gamma$ as connection. Similarly, $T_{AB}{}^{\overline{\gamma}}$ and $\widetilde{R}_{AB\overline{\gamma}}{}^{\overline{\delta}}$ are the torsion and the curvature using $\widetilde{\Omega}_{A\overline{\beta}}{}^{\overline{\gamma}}$ as connection.
The nilpotence of the BRST charge $\widetilde{Q}$ leads to the constraints

$$\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}H_{\overline{\alpha}\overline{\beta}A} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}R_{\overline{\alpha}\overline{\beta}\gamma}{}^\delta = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{\lambda}^{\overline{\gamma}}\widetilde{R}_{\overline{\alpha}\overline{\beta}\overline{\gamma}}{}^{\overline{\delta}} = 0, \tag{3.8}$$

$$\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}T_{\overline{\alpha}\overline{\beta}}{}^a = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}T_{\overline{\alpha}\overline{\beta}}{}^\gamma = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{T}_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0. \tag{3.9}$$

Finally, the anticommutation between Q and $\widetilde{Q}$ determines

$$H_{\alpha\overline{\beta}A} = T_{\alpha\overline{\beta}}{}^a = T_{\alpha\overline{\beta}}{}^\gamma = T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} = \lambda^\alpha\lambda^\beta R_{\overline{\gamma}\alpha\beta}{}^\delta = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{R}_{\gamma\overline{\alpha}\overline{\beta}}{}^{\overline{\delta}} = 0. \tag{3.10}$$

Note that given the decomposition (3.3) for the connections, we can respectively write

$$R_{DC\alpha}{}^\beta = R_{DC}\delta_\alpha{}^\beta + \frac{1}{4}R_{DCef}(\gamma^{ef})_\alpha{}^\beta, \tag{3.11}$$

$$\widetilde{R}_{DC\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{R}_{DC}\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{R}_{DCef}(\gamma^{ef})_{\overline{\alpha}}{}^{\overline{\beta}}. \tag{3.12}$$

3.1.2 Holomorphicity

The holomorphicity of j_B and the antiholomorphicity of $\widetilde{j}_B$ constraints are determined after the use of the equations of motion derived from the action (3.1). The equation for the pure spinor ghosts are

$$\overline{\nabla}\lambda^\alpha + \lambda^\beta(\widetilde{d}_{\overline{\gamma}}C_\beta{}^{\alpha\overline{\gamma}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}S_{\beta\overline{\alpha}}{}^{\alpha\overline{\beta}}) = 0, \quad \overline{\nabla}\omega_\alpha - (\widetilde{d}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}})\omega_\beta = 0, \tag{3.13}$$

and

$$\nabla\widetilde{\lambda}^{\overline{\alpha}} + \widetilde{\lambda}^{\overline{\beta}}(d_\gamma\widetilde{C}_{\overline{\beta}}{}^{\overline{\alpha}\gamma} + \lambda^\alpha\omega_\beta S_{\alpha\overline{\beta}}{}^{\beta\overline{\alpha}}) = 0, \quad \nabla\widetilde{\omega}_{\overline{\alpha}} - (d_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + \lambda^\alpha\omega_\beta S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}})\widetilde{\omega}_{\overline{\beta}} = 0, \tag{3.14}$$

where ∇ is a covariant derivative which acts with Ω or $\widetilde{\Omega}$ connections according to the index structure of the fields it is acting on. For example,

$$\nabla P^{\alpha\overline{\beta}} = \partial P^{\alpha\overline{\beta}} + P^{\gamma\overline{\beta}}\Omega_\gamma{}^\alpha + P^{\alpha\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}{}^{\overline{\beta}}.$$

The variations respect to d_α and $\widetilde{d}_{\overline{\alpha}}$ provide the equations

$$\overline{\Pi}^\alpha + \widetilde{d}_{\overline{\beta}}P^{\alpha\overline{\beta}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\alpha} = 0, \quad \Pi^{\overline{\alpha}} - d_\beta P^{\beta\overline{\alpha}} + \lambda^\alpha\omega_\beta C_\alpha{}^{\beta\overline{\alpha}} = 0. \tag{3.15}$$

The most difficult equations to obtain are those coming from the variation of the superspace coordinates. Let us define $\sigma^A = \delta Z^M E_M{}^A$, then it is not difficult to obtain

$$\delta\Pi^A = \partial\sigma^A - \sigma^B\Pi^C E_B{}^M E_C{}^N \partial_{[N}E_{M]}{}^A(-1)^{C(B+M)}.$$

Here we can express this variation in terms of the connection Ω . In fact,

$$\delta\Pi^A = \nabla\sigma^A - \sigma^B\Pi^C(T_{CB}{}^A + \Omega_{BC}{}^A(-1)^{BC}).$$

There is a point about our notation for the torsion that we should make clear. Using tangent superspace indices, the torsion can be written as

$$T_{BC}{}^A = -E_B{}^N(\partial_N E_C{}^M)E_M{}^A + (-)^{BC}E_C{}^N(\partial_N E_B{}^M)E_M{}^A + \Omega_{BC}{}^A - (-)^{BC}\Omega_{CB}{}^A. \tag{3.16}$$

In our notation, $T_{BC}{}^\alpha$ will mean that the connection in (3.16) is $\Omega_{C\beta}{}^\alpha$ while $T_{BC}{}^{\overline\alpha}$ means that the connection in (3.16) is $\widetilde\Omega_{C\overline\beta}{}^{\overline\alpha}$. Since we also have two connections with bosonic tangent space index $\Omega_{Cb}{}^a$ and $\widetilde\Omega_{Cb}{}^a$, we use $T_{BC}{}^a$ to denote the torsion when we use the first and $\widetilde T_{BC}{}^a$ to denote the torsion when we use the second.

We vary the action (3.1) under these transformations and, after using the equations (3.14), (3.15) and some of the nilpotence constraints, we obtain

$$
\begin{aligned}
\overline\nabla d_\alpha =\ & -\frac{1}{2}\Pi^a\overline\Pi^b(T_{\alpha(ba)} + H_{\alpha ba}) + \frac{1}{2}\Pi^\beta\overline\Pi^a(T_{\beta\alpha a} - H_{\beta\alpha a}) - d_\beta\overline\Pi^a T_{a\alpha}{}^\beta \qquad (3.17)\\
& -\widetilde d_{\overline\beta}\Pi^a(T_{a\alpha}{}^{\overline\beta} + \frac{1}{2}P^{\gamma\overline\beta}(T_{\gamma\alpha a} + H_{\gamma\alpha a})) + \lambda^\beta\omega_\gamma\overline\Pi^a R_{a\alpha\beta}{}^\gamma\\
& +\widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}\Pi^a(\widetilde R_{a\alpha\overline\beta}{}^{\overline\gamma} - \frac{1}{2}\widetilde C_{\overline\beta}{}^{\overline\gamma\delta}(T_{\delta\alpha a} + H_{\delta\alpha a})) - \widetilde d_{\overline\beta}\Pi^\gamma(T_{\gamma\alpha}{}^{\overline\beta} + \frac{1}{2}P^{\delta\overline\beta}H_{\delta\gamma\alpha})\\
& +\lambda^\beta\omega_\gamma\overline\Pi^{\overline\delta}R_{\overline\delta\alpha\beta}{}^\gamma + \widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}\Pi^\delta(\widetilde R_{\delta\alpha\overline\beta}{}^{\overline\gamma} + \frac{1}{2}\widetilde C_{\overline\beta}{}^{\overline\gamma\rho}H_{\rho\delta\alpha})\\
& +d_\beta\widetilde d_{\overline\gamma}(P^{\delta\overline\gamma}T_{\delta\alpha}{}^\beta - \nabla_\alpha P^{\beta\overline\gamma}) + \widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}d_\delta(\nabla_\alpha\widetilde C_{\overline\beta}{}^{\overline\gamma\delta} + \widetilde C_{\overline\beta}{}^{\overline\gamma\rho}T_{\rho\alpha}{}^\delta + P^{\delta\overline\rho}\widetilde R_{\overline\rho\alpha\overline\beta}{}^{\overline\gamma})\\
& +\lambda^\beta\omega_\gamma\widetilde d_{\overline\delta}(\nabla_\alpha C_\beta{}^{\gamma\overline\delta} - P^{\rho\overline\delta}R_{\rho\alpha\beta}{}^\gamma) - \lambda^\beta\omega_\gamma\widetilde\lambda^{\overline\delta}\widetilde\omega_{\overline\rho}(\nabla_\alpha S_{\beta\overline\delta}{}^{\gamma\overline\rho} + C_\beta{}^{\gamma\overline\sigma}\widetilde R_{\overline\sigma\alpha\overline\delta}{}^{\overline\rho}\\
& +\widetilde C_{\overline\delta}{}^{\overline\rho\sigma}R_{\sigma\alpha\beta}{}^\gamma),
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla\widetilde d_{\overline\alpha} =\ & -\frac{1}{2}\Pi^a\overline\Pi^b(T_{\overline\alpha(ba)} + H_{\overline\alpha ba}) + \frac{1}{2}\Pi^a\overline\Pi^{\overline\beta}(T_{\overline\beta\overline\alpha a} + H_{\overline\beta\overline\alpha a}) - \widetilde d_{\overline\beta}\Pi^a T_{a\overline\alpha}{}^{\overline\beta} \qquad (3.18)\\
& -d_\beta\overline\Pi^a(T_{a\overline\alpha}{}^\beta - \frac{1}{2}P^{\beta\overline\gamma}(T_{\overline\gamma\overline\alpha a} - H_{\overline\gamma\overline\alpha a})) + \widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}\Pi^a\widetilde R_{a\overline\alpha\overline\beta}{}^{\overline\gamma}\\
& +\lambda^\beta\omega_\gamma\overline\Pi^a(R_{a\overline\alpha\beta}{}^\gamma - \frac{1}{2}C_\beta{}^{\gamma\overline\delta}(T_{\overline\delta\overline\alpha a} - H_{\overline\delta\overline\alpha a})) - d_\beta\overline\Pi^{\overline\gamma}(T_{\overline\gamma\overline\alpha}{}^\beta + \frac{1}{2}P^{\beta\overline\delta}H_{\overline\gamma\overline\delta\overline\alpha})\\
& +\widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}\Pi^\delta\widetilde R_{\delta\overline\alpha\overline\beta}{}^{\overline\gamma} + \lambda^\beta\omega_\gamma\overline\Pi^{\overline\delta}(R_{\overline\delta\overline\alpha\beta}{}^\gamma + \frac{1}{2}C_\beta{}^{\gamma\overline\rho}H_{\overline\delta\overline\rho\overline\alpha})\\
& +d_\beta\widetilde d_{\overline\gamma}(P^{\beta\overline\delta}T_{\overline\delta\overline\alpha}{}^{\overline\gamma} - \nabla_{\overline\alpha}P^{\beta\overline\gamma}) + \lambda^\beta\omega_\gamma\widetilde d_{\overline\delta}(\nabla_{\overline\alpha}C_\beta{}^{\gamma\overline\delta} + C_\beta{}^{\gamma\overline\rho}\widetilde T_{\overline\rho\overline\alpha}{}^{\overline\delta} - P^{\rho\overline\delta}R_{\rho\overline\alpha\beta}{}^\gamma)\\
& +\widetilde\lambda^{\overline\beta}\widetilde\omega_{\overline\gamma}d_\delta(\nabla_{\overline\alpha}\widetilde C_{\overline\beta}{}^{\overline\gamma\delta} + P^{\delta\overline\rho}\widetilde R_{\overline\rho\overline\alpha\overline\beta}{}^{\overline\gamma}) - \lambda^\beta\omega_\gamma\widetilde\lambda^{\overline\delta}\widetilde\omega_{\overline\rho}(\nabla_{\overline\alpha}S_{\beta\overline\delta}{}^{\gamma\overline\rho} + C_\beta{}^{\gamma\overline\sigma}\widetilde R_{\overline\sigma\overline\alpha\overline\delta}{}^{\overline\rho}\\
& +\widetilde C_{\overline\delta}{}^{\overline\rho\sigma}R_{\sigma\overline\alpha\beta}{}^\gamma).
\end{aligned}
$$

From these equations, (3.13), (3.14) and also two equations in (3.10) we obtain the holomorphicity constraints. In fact, $\overline\nabla j_B = 0$ implies

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$$T_{\alpha(ab)} = H_{\alpha ab} = T_{\alpha\beta a} - H_{\alpha\beta a} = T_{a\alpha}{}^{\beta} = T_{a\alpha}{}^{\overline{\beta}} + P^{\gamma\overline{\beta}}T_{\gamma\alpha a} = \lambda^{\alpha}\lambda^{\beta}R_{a\alpha\beta}{}^{\gamma} = 0,$$

$$\widetilde{R}_{a\alpha\overline{\beta}}{}^{\overline{\gamma}} - \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}T_{\delta\alpha a} = T_{\gamma\alpha}{}^{\overline{\beta}} + \frac{1}{2}P^{\delta\overline{\beta}}H_{\delta\gamma\alpha} = \widetilde{R}_{\delta\alpha\overline{\beta}}{}^{\overline{\gamma}} + \frac{1}{2}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho}H_{\rho\delta\alpha} = 0, \qquad (3.19)$$

$$P^{\delta\overline{\gamma}}T_{\delta\alpha}{}^{\beta} - \nabla_{\alpha}P^{\beta\overline{\gamma}} - C_{\alpha}{}^{\beta\overline{\gamma}} = \nabla_{\alpha}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho}T_{\rho\alpha}{}^{\delta} + P^{\delta\overline{\rho}}\widetilde{R}_{\overline{\rho}\alpha\overline{\beta}}{}^{\overline{\gamma}} - S_{\alpha\beta}{}^{\delta\overline{\gamma}} = 0,$$

$$\lambda^{\alpha}\lambda^{\beta}(\nabla_{\alpha}C_{\beta}{}^{\gamma\overline{\delta}} - P^{\rho\overline{\delta}}R_{\rho\alpha\beta}{}^{\gamma}) = \lambda^{\alpha}\lambda^{\beta}(\nabla_{\alpha}S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}} + C_{\beta}{}^{\gamma\overline{\sigma}}\widetilde{R}_{\overline{\sigma}\alpha\overline{\delta}}{}^{\overline{\rho}} + \widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}R_{\sigma\alpha\beta}{}^{\gamma}) = 0,$$

and $\nabla\widetilde{j}_B = 0$ implies

$$T_{\overline{\alpha}(ab)} = H_{\overline{\alpha}ab} = T_{\overline{\alpha}\overline{\beta}a} + H_{\overline{\alpha}\overline{\beta}a} = T_{a\overline{\alpha}}{}^{\overline{\beta}} = T_{a\overline{\alpha}}{}^{\beta} - P^{\beta\overline{\gamma}}T_{\overline{\gamma}\overline{\alpha}a} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{R}_{a\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0,$$

$$R_{a\overline{\alpha}\beta}{}^{\gamma} - C_{\beta}{}^{\gamma\overline{\delta}}T_{\overline{\delta}\overline{\alpha}a} = T_{\overline{\gamma}\overline{\alpha}}{}^{\beta} + \frac{1}{2}P^{\beta\overline{\delta}}H_{\overline{\gamma}\overline{\delta}\overline{\alpha}} = R_{\overline{\delta}\overline{\alpha}\beta}{}^{\gamma} + \frac{1}{2}C_{\beta}{}^{\gamma\overline{\rho}}H_{\overline{\delta}\overline{\rho}\overline{\alpha}} = 0 \qquad (3.20)$$

$$P^{\beta\overline{\delta}}T_{\overline{\delta}\overline{\alpha}}{}^{\overline{\gamma}} - \nabla_{\overline{\alpha}}P^{\beta\overline{\gamma}} + \widetilde{C}_{\overline{\alpha}}{}^{\overline{\gamma}\beta} = \nabla_{\overline{\alpha}}C_{\beta}{}^{\gamma\overline{\delta}} + C_{\beta}{}^{\gamma\overline{\rho}}T_{\overline{\rho}\overline{\alpha}}{}^{\overline{\delta}} - P^{\rho\overline{\delta}}R_{\rho\overline{\alpha}\beta}{}^{\gamma} - S_{\beta\overline{\alpha}}{}^{\gamma\overline{\delta}} = 0$$

$$\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}(\nabla_{\overline{\alpha}}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta} + P^{\delta\overline{\rho}}\widetilde{R}_{\overline{\rho}\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}) = 0, \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}(\nabla_{\overline{\alpha}}S_{\delta\overline{\beta}}{}^{\rho\overline{\gamma}} + C_{\delta}{}^{\rho\overline{\sigma}}\widetilde{R}_{\overline{\sigma}\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\sigma}R_{\sigma\overline{\alpha}\delta}{}^{\rho}) = 0.$$

3.1.3 Solving the Bianchi identities

We can gauge-fix some of the torsion components and determine others through the use of Bianchi identities. It is not necessary but it will simplify the computation of the one-loop beta functions. As in [24], we can set $H_{\alpha\beta\gamma} = H_{\alpha\beta\overline{\gamma}} = H_{\alpha\overline{\beta}\overline{\gamma}} = H_{\overline{\alpha}\overline{\beta}\overline{\gamma}} = 0$ since there is no such ten-dimensional superfields satisfying the nilpotency constrains of Q and $\widetilde{Q}$. We can use the Lorentz rotations to gauge fix $T_{\alpha\beta}{}^{a} = \gamma_{\alpha\beta}^{a}$ and $T_{\overline{\alpha}\overline{\beta}}{}^{a} = \gamma_{\overline{\alpha}\overline{\beta}}^{a}$, therefore the above constraints imply $H_{\alpha\beta a} = (\gamma_a)_{\alpha\beta}$ and $H_{\overline{\alpha}\overline{\beta}a} = -(\gamma_a)_{\overline{\alpha}\overline{\beta}}$. We can use the shift symmetry of the action (3.1)

$$\delta d_{\alpha} = \delta\Omega_{\alpha\beta}{}^{\gamma}\lambda^{\beta}\omega_{\gamma}, \quad \delta\widetilde{d}_{\overline{\alpha}} = \delta\widetilde{\Omega}_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}, \quad \delta C_{\alpha}{}^{\beta\overline{\gamma}} = P^{\delta\overline{\gamma}}\delta\Omega_{\delta\alpha}{}^{\beta}, \quad \delta\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} = -P^{\gamma\overline{\delta}}\delta\widetilde{\Omega}_{\overline{\delta}\overline{\alpha}}{}^{\overline{\beta}},$$

$$\delta S_{\alpha\overline{\beta}}{}^{\gamma\overline{\delta}} = C_{\alpha}{}^{\gamma\overline{\rho}}\delta\widetilde{\Omega}_{\overline{\rho}\overline{\beta}}{}^{\overline{\delta}} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\delta}\rho}\delta\Omega_{\rho\alpha}{}^{\gamma},$$

to gauge-fix $T_{\alpha\beta}{}^{\gamma} = T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0$.

The Bianchi identity for the torsion is

$$(\nabla T)_{ABC}{}^{D} \equiv \nabla_{[A}T_{BC]}{}^{D} + T_{[AB}{}^{E}T_{EC]}{}^{D} - R_{[ABC]}{}^{D} = 0, \qquad (3.21)$$

where brackets in (3.21) mean (anti-)symmetrization respect to the ABC indices. The curvature will be R or $\widetilde{R}$ if the upper index D is δ or $\overline{\delta}$ respectively. When $D = d$, we use the notation $(\nabla T)_{ABC}{}^{d}$ or $(\nabla\widetilde{T})_{ABC}{}^{d}$, if we use the connection $\Omega_{Bc}{}^{a}$ or $\widetilde{\Omega}_{Bc}{}^{a}$; then the curvatures in each case will be R or $\widetilde{R}$.

The Bianchi identity $(\nabla T)_{\alpha\beta\gamma}{}^a = 0$ implies $T_{\alpha ab} = 2(\gamma_{ab})_\alpha{}^\beta \Omega_\beta$. Similarly, the Bianchi identity $(\nabla \widetilde{T})_{\overline{\alpha}\overline{\beta}\overline{\gamma}}{}^a = 0$ implies $\widetilde{T}_{\overline{\alpha} ab} = 2(\gamma_{ab})_{\overline{\alpha}}{}^{\overline{\beta}} \Omega_{\overline{\beta}}$. The Bianchi identity $(\nabla T)_{\alpha\overline{\beta}\overline{\gamma}}{}^a = 0$ implies $\widetilde{\Omega}_\alpha = \widetilde{T}_{\alpha a}{}^b = 0$. Similarly, the Bianchi identity $(\nabla T)_{\overline{\alpha}\beta\gamma}{}^a = 0$ implies $\Omega_{\overline{\alpha}} = T_{\overline{\alpha} a}{}^b = 0$. It is not difficult to show that the constraints $T_{a\alpha}{}^\alpha = T_{a\overline{\alpha}}{}^{\overline{\alpha}} = 0$ imply $\Omega_a = \widetilde{\Omega}_a = 0$.

We can write two sets of Bianchi identities for H depending on what is the connection we choose in the covariant derivative. Note that the components of the superfield H do not depend on such choice. The Bianchi identities come from $\nabla H = 0$ and $\widetilde{\nabla} H = 0$ and it is not difficult to check that both sets are equivalent. Let us write only one of them

$$(\nabla H)_{ABCD} \equiv \nabla_{[A} H_{BCD]} + \frac{3}{2} T_{[AB}{}^E H_{ECD]} = 0. \tag{3.22}$$

There is one more Bianchi identity involving a derivative of the curvature

$$(\nabla R)_{ABCD}{}^E \equiv \nabla R_{[ABC]D}{}^E + T_{[AB}{}^F R_{FC]D}{}^E = 0. \tag{3.23}$$

The identities $(\nabla H)_{\alpha\beta\gamma\delta}, (\nabla H)_{\alpha\beta\gamma\overline{\delta}}, (\nabla H)_{\alpha\beta\overline{\gamma}\overline{\delta}}, (\nabla H)_{\alpha\overline{\beta}\overline{\gamma}\overline{\delta}}, (\nabla H)_{\overline{\alpha}\overline{\beta}\overline{\gamma}\overline{\delta}}$ are easily satisfied if we recall the identities for gamma matrices $\gamma^a_{(\alpha\beta}(\gamma_a)_{\gamma)\delta} = \gamma^a_{(\overline{\alpha}\overline{\beta}}(\gamma_a)_{\overline{\gamma})\overline{\delta}} = 0$. The identities $(\nabla H)_{a\alpha\beta\gamma}, (\nabla H)_{a\alpha\beta\overline{\gamma}}, (\nabla H)_{a\alpha\overline{\beta}\overline{\gamma}}, (\nabla H)_{a\overline{\alpha}\overline{\beta}\overline{\gamma}}$ are satisfied after using the dimension-$\frac{1}{2}$ constraints. The identity $(\nabla H)_{ab\alpha\beta} = 0$ implies $T_{abc} + H_{abc} = 0$ and the identity $(\widetilde{\nabla} H)_{ab\overline{\alpha}\overline{\beta}} = 0$ implies $\widetilde{T}_{abc} - H_{abc} = 0$. The identity $(\nabla H)_{ab\alpha\overline{\beta}} = 0$ is satisfied if we use the constraints involving the superfield $P^{\alpha\overline{\beta}}$ in the first lines of (3.19) and (3.20).

3.1.4 The remaining equation of motion

In the computation of the one-loop beta function we will need to know the equation of motion for Π^a and $\overline{\Pi}^a$. Since we know that the difference $\nabla\overline{\Pi}^a - \overline{\nabla}\Pi^a$ is given by the torsion components, then we only need to determine $\nabla\overline{\Pi}^a + \overline{\nabla}\Pi^a$ which is determined by the varying the action respect to $\sigma^a = \delta Z^M E_M{}^a$. To make life simpler we will write this equation using the above results for torsion and H components. The equation turns out to be

$$\frac{1}{2}(\widetilde{\nabla}\overline{\Pi}_a + \overline{\nabla}\Pi_a) = \frac{1}{2}\Pi^b\overline{\Pi}^c H_{cba} - \frac{1}{2}\Pi^\alpha\overline{\Pi}^b T_{\alpha ab} + d_\alpha\overline{\Pi}^b T_{ab}{}^\alpha + \lambda^\alpha\omega_\beta\overline{\Pi}^b R_{ab\alpha}{}^\beta$$
$$+\frac{1}{2}\widetilde{d}_{\overline{\alpha}}\Pi^\beta T_{a\beta}{}^{\overline{\alpha}} + \widetilde{d}_{\overline{\alpha}}\Pi^b(T_{ab}{}^{\overline{\alpha}} + \frac{1}{2}P^{\beta\overline{\alpha}}T_{\beta ab})$$
$$+\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^b(\widetilde{R}_{ab\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{2}\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}T_{\gamma ab})$$
$$+\frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^\gamma\widetilde{R}_{a\gamma\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{2}d_\alpha\overline{\Pi}^{\overline{\beta}}T_{a\overline{\beta}}{}^\alpha + \frac{1}{2}\lambda^\alpha\omega_\beta\overline{\Pi}^{\overline{\gamma}}R_{a\overline{\gamma}\alpha}{}^\beta + d_\alpha\widetilde{d}_{\overline{\beta}}\nabla_a P^{\alpha\overline{\beta}}$$
$$+\lambda^\alpha\omega_\beta\widetilde{d}_{\overline{\gamma}}(\nabla_a C_\alpha{}^{\beta\overline{\gamma}} - P^{\delta\overline{\gamma}}R_{a\delta\alpha}{}^\beta) + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}d_\gamma(\nabla_a\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + P^{\gamma\overline{\delta}}\widetilde{R}_{a\overline{\delta}\overline{\alpha}}{}^{\overline{\beta}})$$
$$+\lambda^\alpha\omega_\beta\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}(\nabla_a S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}} - \widetilde{C}_{\overline{\gamma}}{}^{\overline{\delta}\rho}R_{a\rho\alpha}{}^\beta - C_\alpha{}^{\beta\overline{\rho}}\widetilde{R}_{a\overline{\rho}\overline{\gamma}}{}^{\overline{\delta}}). \qquad (3.24)$$

3.1.5 Ghost number conservation

As it was shown in [24], the vanishing of the ghost number anomaly determines that the spinorial derivatives of the dilaton superfield Φ are proportional to the scale connection Ω. This relation is crucial to cancel the beta function in heterotic string case [25] and will be equally essential in our computation. Let us recall how this relation is obtained. Consider the coupling between ghost number currents and the connections in the action (3.1). Namely

$$\frac{1}{2\pi\alpha'}\int d^2z\;(J\overline{\Omega} + \widetilde{J}\Omega).$$

The BRST variation on this term contains the term

$$-\frac{1}{2\pi\alpha'}\int d^2z\;(\overline{\partial}J\lambda^\alpha\Omega_\alpha + \partial\widetilde{J}\lambda^\alpha\widetilde{\Omega}_\alpha).$$

The anomaly in the ghost number current conservation turns out to be proportional to the two dimensional Ricci scalar, as noted by dimensional grounds. The proportionality can be determined by performing a Weyl transformation, around the flat world-sheet, of the anomaly equation. In this way, the triple-pole in the OPE between the current and the corresponding stress tensor yields

$$\nabla_\alpha\Phi = 4\Omega_\alpha, \quad \nabla_{\overline{\alpha}}\Phi = 4\widetilde{\Omega}_{\overline{\alpha}}, \qquad (3.25)$$

which will be used in section 5 to cancel the UV divergent part of the effective action.

3.2 Covariant Background Field Expansion

We use the method explained in [20] and [25]. Here, we need to define a straight-line geodesic which joins a point in superspace to neighbor ones and allows us to perform

an expansion in superspace. It is given by Y^A which satisfies the geodesic equation $\Delta Y^A = Y^B \nabla_B Y^A = 0$. The connection we choose to define this covariant derivative has the non-vanishing components $\Omega_{Aa}{}^b, \Omega_{A\alpha}{}^\beta$ and $\widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}}$. These same connections are defined in the action (3.1). In this way, the covariant expansions of the different objects in (3.1) are determined by

$$\Delta\Pi^A = \nabla Y^A - Y^B \Pi^C T_{CB}{}^A, \quad \Delta\overline{\Omega}_\alpha{}^\beta = -Y^A \overline{\Pi}{}^B R_{BA\alpha}{}^\beta, \quad \Delta\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} = -Y^A \Pi^B \widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}}.$$
(3.26)

Any superfield Ψ is expanded as $\Delta\Psi = Y^A \nabla_A \Psi$.

As in [25], we see that $d_\alpha, \widetilde{d}_{\overline{\alpha}}$ and the pure spinor ghosts are treated as fundamental fields, then we expand them according to

$$d_\alpha = d_{\alpha 0} + \widehat{d}_\alpha, \quad \lambda^\alpha = \lambda^\alpha_0 + \widehat{\lambda}^\alpha, \quad \omega_\alpha = \omega_{\alpha 0} + \widehat{\omega}_\alpha, \tag{3.27}$$

$$\widetilde{d}_{\overline{\alpha}} = \widetilde{d}_{\overline{\alpha}0} + \widehat{\widetilde{d}}_{\overline{\alpha}}, \quad \widetilde{\lambda}^{\overline{\alpha}} = \widetilde{\lambda}^{\overline{\alpha}}_0 + \widehat{\widetilde{\lambda}}^{\overline{\alpha}}, \quad \widetilde{\omega}_{\overline{\alpha}} = \widetilde{\omega}_{\overline{\alpha}0} + \widehat{\widetilde{\omega}}_{\overline{\alpha}0}, \tag{3.28}$$

where the subindex 0 means the background value of the corresponding field which will dropped in the subsequent discussion.

The quadratic part of the expansion of (3.1), excluding the Fradkin-Tseytlin term, has the form

$$S_2 = S_p + \frac{1}{2\pi\alpha'} \int d^2z \; (Y^A Y^B E_{BA} + Y^A \overline{\nabla} Y^B C_{BA} + Y^A \nabla Y^B \overline{C}_{BA} \tag{3.29}$$

$$+\widehat{d}_\alpha Y^A \overline{D}_A{}^\alpha + \widehat{\widetilde{d}}_{\overline{\alpha}} Y^A D_A{}^{\overline{\alpha}} + (\widehat{\lambda}^\alpha \widehat{\omega}_\beta)\overline{H}_\alpha{}^\beta + (\widehat{\widetilde{\lambda}}^{\overline{\alpha}} \widehat{\widetilde{\omega}}_{\overline{\beta}})H_{\overline{\alpha}}{}^{\overline{\beta}} + (\widehat{\lambda}^\alpha \omega_\beta + \lambda^\alpha \widehat{\omega}_\beta) Y^A \overline{I}_{A\alpha}{}^\beta$$

$$+(\widehat{\widetilde{\lambda}}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\beta}} + \widetilde{\lambda}^{\overline{\alpha}} \widehat{\widetilde{\omega}}_{\overline{\beta}}) Y^A I_{A\overline{\alpha}}{}^{\overline{\beta}} + \widehat{d}_\alpha \widehat{\widetilde{d}}_{\overline{\beta}} P^{\alpha\overline{\beta}} + (\widehat{\lambda}^\alpha \omega_\beta + \lambda^\alpha \widehat{\omega}_\beta)\widehat{\widetilde{d}}_{\overline{\gamma}} C_\alpha{}^{\beta\overline{\gamma}} +$$

$$(\widehat{\widetilde{\lambda}}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\beta}} + \widetilde{\lambda}^{\overline{\alpha}} \widehat{\widetilde{\omega}}_{\overline{\beta}})\widehat{d}_\gamma \widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + (\widehat{\lambda}^\alpha \omega_\beta + \lambda^\alpha \widehat{\omega}_\beta)(\widehat{\widetilde{\lambda}}^{\overline{\gamma}} \widetilde{\omega}_{\overline{\delta}} + \widetilde{\lambda}^{\overline{\gamma}} \widehat{\widetilde{\omega}}_{\overline{\delta}})S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}}),$$

where $E_{BA}, C_{BA}, \ldots$ are background superfields given by

$$
\begin{aligned}
E_{BA} &= \frac{1}{4}\Pi^C\overline{\Pi}^D(T_{CB}{}^E H_{EDA}(-1)^{D(C+B)} - T_{DB}{}^E H_{ECA}(-1)^{BC} + \nabla_B H_{DCA}(-1)^{B(C+D)} \\
&\quad + 2T_{CB}{}^a T_{DAa}(-1)^{D(C+B)}) - \frac{1}{4}\Pi^{(a}\overline{\Pi}^{C)}(R_{CBAa} - T_{CB}{}^D T_{DAa} + \nabla_B T_{CAa}(-1)^{BC}) \\
&\quad + \frac{1}{2}d_\alpha\overline{\Pi}^C(-1)^{A+B}(-R_{CBA}{}^\alpha + T_{CB}{}^D T_{DA}{}^\alpha - \nabla_B T_{CA}{}^\alpha(-1)^{BC}) \qquad (3.30) \\
&\quad + \frac{1}{2}\widetilde{d}_{\overline{\alpha}}\Pi^C(-1)^{A+B}(-R_{CBA}{}^{\overline{\alpha}} + T_{CB}{}^D T_{DA}{}^{\overline{\alpha}} - \nabla_B T_{CA}{}^{\overline{\alpha}}(-1)^{BC}) \\
&\quad + \frac{1}{2}\lambda^\alpha\omega_\beta\overline{\Pi}^C(T_{CB}{}^D R_{DA\alpha}{}^\beta - \nabla_B R_{CA\alpha}{}^\beta(-1)^{BC}) \\
&\quad + \frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^C(T_{CB}{}^D R_{DA\overline{\alpha}}{}^{\overline{\beta}} - \nabla_B R_{CA\overline{\alpha}}{}^{\overline{\beta}}(-1)^{BC}) + \frac{1}{2}d_\alpha\widetilde{d}_{\overline{\beta}}\nabla_B\nabla_A P^{\alpha\overline{\beta}} \\
&\quad + \frac{1}{2}\lambda^\alpha\omega_\beta\widetilde{d}_{\overline{\gamma}}\nabla_B\nabla_A C_\alpha{}^{\beta\overline{\gamma}}(-1)^{A+B} + \frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}d_\gamma\nabla_B\nabla_A \widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}(-1)^{A+B} \\
&\quad + \frac{1}{2}\lambda^\alpha\omega_\beta\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}\nabla_B\nabla_A S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}},
\end{aligned}
$$

$$
C_{BA} = -\frac{1}{4}\Pi^a T_{BAa} - \frac{1}{2}\Pi^C T_{CAa}\delta_B^a - \frac{1}{4}\Pi^A H_{CBA} - \frac{1}{2}d_\alpha T_{BA}{}^\alpha(-1)^{A+B} - \frac{1}{2}\lambda^\alpha\omega_\beta R_{BA\alpha}{}^\beta,
\tag{3.31}
$$

$$
\overline{C}_{BA} = -\frac{1}{4}\Pi^b T_{BAa} - \frac{1}{2}\overline{\Pi}^C T_{CAa}\delta_B^a + \frac{1}{4}\Pi^A H_{CBA} - \frac{1}{2}\widetilde{d}_{\overline{\alpha}}\widetilde{T}_{BA}{}^{\overline{\alpha}}(-1)^{A+B} - \frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}},
\tag{3.32}
$$

$$
\overline{D}_A{}^\alpha = -\overline{\Pi}^B T_{BA}{}^\alpha + \widetilde{d}_{\overline{\beta}}\nabla_A P^{\alpha\overline{\beta}}(-1)^A + \widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\nabla_A\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\alpha},
\tag{3.33}
$$

$$
D_A{}^{\overline{\alpha}} = -\Pi^B\widetilde{T}_{BA}{}^{\overline{\alpha}} - d_\beta\nabla_A P^{\beta\overline{\alpha}}(-1)^A + \lambda^\beta\omega_\gamma\nabla_A C_\beta{}^{\gamma\overline{\alpha}},
\tag{3.34}
$$

$$
\overline{H}_\alpha{}^\beta = \overline{\Omega}_\alpha{}^\beta + \widetilde{d}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}}\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}},
\tag{3.35}
$$

$$
H_{\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} + d_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + \lambda^\gamma\omega_\delta S_{\gamma\overline{\alpha}}{}^{\delta\overline{\beta}},
\tag{3.36}
$$

$$
\overline{I}_{A\alpha}{}^\beta = -\overline{\Pi}^A R_{BA\alpha}{}^\beta + \widetilde{d}_{\overline{\gamma}}\nabla_A C_\alpha{}^{\beta\overline{\gamma}}(-1)^A + \widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}\nabla_A S_{\alpha\overline{\gamma}}{}^{\beta\overline{\gamma}},
\tag{3.37}
$$

$$
I_{A\overline{\alpha}}{}^{\overline{\beta}} = -\Pi^B\widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}} + d_\gamma\nabla_A\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}(-1)^A + \lambda^\gamma\omega_\delta\nabla_A S_{\gamma\overline{\alpha}}{}^{\delta\overline{\beta}}.
\tag{3.38}
$$

In (3.30) S_p provides the propagators for the quantum fields and is given by

$$
S_p = \frac{1}{2\pi\alpha'}\int d^2z\,(\frac{1}{2}\nabla Y^a\overline{\nabla}Y_a + \widehat{d}_\alpha\overline{\nabla}Y^\alpha + \widehat{\widetilde{d}}_{\overline{\alpha}}\nabla Y^{\overline{\alpha}}) + \mathcal{L}_{pure},
\tag{3.39}
$$

where $\mathcal{L}_{pure}$ is the Lagrangian for the pure spinor ghosts.

3.3 The one-loop UV divergent Part of the Effective Action

The effective action is given by

$$e^{-S_{eff}} = \int D\mathcal{Q}\, e^{-S}, \tag{3.40}$$

where $\mathcal{Q}$ represents the quantum fluctuations.

To compute the one-loop beta functions we need to expand (3.1) up to second order in the quantum fields. In this way, we will obtain the UV divergent part[2] of the effective action, S_Λ. Here Λ is UV scale. Note that the Fradkin-Tseytlin term is evaluated on a sphere with metric $\Lambda dz d\bar{z}$. Finally, the complete UV divergent part of the effective action becomes

$$S_\Lambda + \frac{1}{2\pi} \int d^2z \; (\nabla \overline{\Pi}^A \nabla_A \Phi + \overline{\Pi}^A \Pi^B \nabla_B \nabla_A \Phi) \log \Lambda. \tag{3.41}$$

The computation of S_Λ is performed by contracting the quantum fields. From (3.39) we read

$$Y^a(z,\bar{z}) Y^b(w,\bar{w}) \to -\alpha' \eta^{ab} \log |z-w|^2 \tag{3.42}$$

$$\widehat{d}_\alpha(z) Y^\beta(w) \to \frac{\alpha' \delta_\alpha^{\ \beta}}{(z-w)}, \quad \widehat{\widetilde{d}}_{\overline{\alpha}}(\bar{z}) Y^{\overline{\beta}}(\bar{w}) \to \frac{\alpha' \delta_{\overline{\alpha}}^{\ \overline{\beta}}}{(\bar{z}-\bar{w})}. \tag{3.43}$$

For the pure spinor ghosts we note that, because of (3.3), they enter in the combinations

$$N^{ab} = \frac{1}{2}(\lambda \gamma^{ab} \omega), \quad J = \lambda^\alpha \omega_\alpha, \quad \widetilde{N}^{ab} = \frac{1}{2}(\widetilde{\lambda} \gamma^{ab} \widetilde{\omega}), \quad \widetilde{J} = \widetilde{\lambda}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\alpha}}.$$

We can expand each of these combinations as $J + J_1 + J_2$, similarly for $\widetilde{J}$, N^{ab} and $\widetilde{N}^{ab}$. As in [25], the only relevant OPE's involving the pure spinor ghosts and contributing to S_Λ are

$$N_1^{ab}(z) N_1^{cd}(w) \to \frac{1}{(z-w)}(-\eta^{a[c} N^{d]b}(w) + \eta^{b[c} N^{d]a}(w)), \tag{3.44}$$

$$\widetilde{N}_1^{ab}(\bar{z}) \widetilde{N}_1^{cd}(\bar{w}) \to \frac{1}{(\bar{z}-\bar{w})}(-\eta^{a[c} \widetilde{N}^{d]b}(\bar{w}) + \eta^{b[c} \widetilde{N}^{d]a}(\bar{w})). \tag{3.45}$$

The one-loop contributions to S_Λ come from self-contraction of Y^A's in the term with E_{BA} in (3.30) and a series of double contractions in (3.30). These come from products between the term involving C_{BA} with the one involving $\overline{C}_{BA}$, C_{BA} with $\overline{D}_A{}^{\overline{\beta}}$, $\overline{C}_{BA}$ with $D_A{}^\beta$, $\overline{D}_A{}^{\overline{\beta}}$ with $D_A{}^\beta$, E_{BA} with $P^{\alpha\overline{\beta}}$, $\overline{I}_{C\alpha}{}^\beta$ with $C_\alpha{}^{\beta\overline{\gamma}}$, $I_{C\overline{\alpha}}{}^{\overline{\beta}}$ with

$\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$ and $S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}}$ with itself. After adding up all these contributions, the one-loop UV divergent part of the effective action is proportional to

$$\int d^2 z \; [-\eta^{ab}E_{ba} + \eta^{a[c}\eta^{d]b}C_{ba}\overline{C}_{dc} + \eta^{ab}C_{[a\alpha]}\overline{D}_b{}^\alpha + \eta^{ab}\overline{C}_{[a\overline{\alpha}]}D_b{}^{\overline{\alpha}} + \overline{D}_{\overline{\alpha}}{}^\beta D_\beta{}^{\overline{\alpha}} + E_{[\alpha\overline{\beta}]}P^{\alpha\overline{\beta}}$$

(3.46)

$$+ N^{ab}\overline{I}_{\overline{\alpha}a}{}^c C_{cb}{}^{\overline{\alpha}} + \widetilde{N}^{ab}I_{\alpha a}{}^c \widetilde{C}_{cb}{}^\alpha + \frac{1}{2}N^{ab}\widetilde{N}^{cd}S_a{}^e{}_c{}^f S_{bedf} + \nabla\overline{\Pi}^A\nabla_A\Phi + \overline{\Pi}^A\Pi^B\nabla_B\nabla_A\Phi] \log\Lambda,$$

where we used the expressions (3.3).

Now it will be shown that (3.46) vanishes as consequence of the classical BRST constraints.

3.4 One-loop Conformal Invariance

To write the equations derived from the vanishing of (3.46), we need to determine $\nabla\overline{\Pi}^A$ from the classical equations of motion from (3.1). In order to do this, we need to know

$$\overline{\nabla}\Pi^A - \nabla\overline{\Pi}^A = \Pi^B\overline{\Pi}^C T_{CB}{}^A.$$

(3.47)

Note that we are using here the connection $\Omega_A{}^B$ to calculate the covariant derivatives and the torsion components.

The equation for $\nabla\overline{\Pi}_a$ is

$$\begin{aligned}
\nabla\overline{\Pi}_a &= \Pi^b\overline{\Pi}^c T_{abc} - \Pi^\alpha\overline{\Pi}^b T_{\alpha ab} + \widetilde{d}_{\overline{\alpha}}\Pi^b T_{ab}{}^{\overline{\alpha}} + d_\alpha\overline{\Pi}^b T_{ab}{}^\alpha + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^b \widetilde{R}_{ab\overline{\alpha}}{}^{\overline{\beta}} \quad (3.48) \\
&\quad + \lambda^\alpha\omega_\beta\overline{\Pi}^b R_{ab\alpha}{}^\beta + \widetilde{d}_{\overline{\alpha}}\Pi^\beta T_{a\beta}{}^{\overline{\alpha}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^\gamma \widetilde{R}_{a\gamma\overline{\alpha}}{}^{\overline{\beta}} + d_\alpha\widetilde{d}_{\overline{\beta}}\nabla_a P^{\alpha\overline{\beta}} \\
&\quad + \lambda^\alpha\omega_\beta\widetilde{d}_{\overline{\gamma}}(\nabla_a C_\alpha{}^{\beta\gamma} - P^{\delta\overline{\gamma}}R_{a\delta\alpha}{}^\beta) + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}d_\gamma(\nabla_a\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + P^{\gamma\overline{\delta}}\widetilde{R}_{a\overline{\delta}\overline{\alpha}}{}^{\overline{\beta}}) \\
&\quad + \lambda^\alpha\omega_\beta\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}(\nabla_a S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}} - \widetilde{C}_{\overline{\gamma}}{}^{\overline{\delta}\rho}R_{a\rho\alpha}{}^\beta - C_\alpha{}^{\beta\overline{\rho}}\widetilde{R}_{a\overline{\rho}\overline{\gamma}}{}^{\overline{\delta}}).
\end{aligned}$$

Now we compute the equation for $\overline{\Pi}^\alpha$. We start by noting that this world-sheet field is determined from the equation of motion (3.15), then

$$\nabla\overline{\Pi}^\alpha = -\nabla(\widetilde{d}_{\overline{\beta}}P^{\alpha\overline{\beta}} + \widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\alpha}).$$

Remember that the covariant derivative on $P^{\alpha\overline{\beta}}$ and $\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$ acts with $\Omega_\alpha{}^\beta$ on α-indices and with $\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}}$ on $\overline{\alpha}$-indices. Now we can use the equations (3.14) and (3.18) to obtain

$$
\begin{aligned}
\nabla\overline{\Pi}^{\alpha} \;=\;& d_{\beta}\widetilde{d}_{\overline{\gamma}}(\widetilde{C}_{\overline{\delta}}{}^{\overline{\gamma}\beta}P^{\alpha\overline{\delta}}+P^{\beta\overline{\delta}}\nabla_{\overline{\delta}}P^{\alpha\overline{\gamma}}) + \lambda^{\beta}\omega_{\gamma}\widetilde{d}_{\overline{\delta}}(-S_{\beta\overline{\rho}}{}^{\gamma\overline{\delta}}P^{\alpha\overline{\rho}}+C_{\beta}{}^{\gamma\overline{\rho}}\nabla_{\overline{\rho}}P^{\alpha\overline{\delta}}) && (3.49)\\
& -\widetilde{d}_{\overline{\beta}}\Pi^{a}\nabla_{a}P^{\alpha\overline{\beta}}-\widetilde{d}_{\overline{\beta}}\Pi^{\gamma}\nabla_{\gamma}P^{\alpha\overline{\beta}}+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}d_{\delta}(\widetilde{C}_{\overline{\beta}}{}^{\overline{\rho}\delta}\widetilde{C}_{\overline{\rho}}{}^{\overline{\gamma}\alpha}-\widetilde{C}_{\overline{\rho}}{}^{\overline{\gamma}\delta}\widetilde{C}_{\overline{\beta}}{}^{\overline{\rho}\alpha}-P^{\delta\overline{\rho}}\nabla_{\overline{\rho}}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\alpha}\\
& -P^{\alpha\overline{\delta}}(\nabla_{\overline{\delta}}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}+P^{\delta\overline{\epsilon}}\widetilde{R}_{\overline{\epsilon}\delta\overline{\beta}}{}^{\overline{\gamma}}))+\lambda^{\beta}\omega_{\gamma}\widetilde{\lambda}^{\overline{\delta}}\widetilde{\omega}_{\overline{\rho}}(S_{\beta\overline{\delta}}{}^{\gamma\overline{\sigma}}\widetilde{C}_{\overline{\sigma}}{}^{\overline{\rho}\alpha}-S_{\beta\overline{\sigma}}{}^{\gamma\overline{\rho}}\widetilde{C}_{\overline{\delta}}{}^{\overline{\sigma}\alpha}\\
& +C_{\beta}{}^{\gamma\overline{\sigma}}\nabla_{\overline{\sigma}}\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\alpha}+P^{\alpha\overline{\epsilon}}(\nabla_{\overline{\epsilon}}S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}}+C_{\beta}{}^{\gamma\overline{\sigma}}\widetilde{R}_{\overline{\sigma}\overline{\epsilon}\overline{\delta}}{}^{\overline{\rho}}+\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}R_{\sigma\overline{\epsilon}\beta}{}^{\gamma}))\\
& -\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^{a}(\nabla_{a}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\alpha}+\widetilde{R}_{a\overline{\delta}\overline{\beta}}{}^{\overline{\gamma}}P^{\alpha\overline{\delta}})-\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^{\delta}S_{\delta\overline{\beta}}{}^{\alpha\overline{\gamma}}.
\end{aligned}
$$

To obtain the equation for $\overline{\Pi}^{\overline{\alpha}}$ we can use (3.47). After all this we get

$$
\begin{aligned}
\nabla\overline{\Pi}^{\overline{\alpha}} \;=\;& d_{\beta}\widetilde{d}_{\overline{\gamma}}(C_{\delta}{}^{\beta\overline{\gamma}}P^{\delta\overline{\alpha}}-P^{\delta\overline{\gamma}}\nabla_{\delta}P^{\beta\overline{\alpha}})+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}d_{\delta}(S_{\rho\overline{\beta}}{}^{\delta\overline{\gamma}}P^{\rho\overline{\alpha}}-\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho}\nabla_{\rho}P^{\delta\overline{\alpha}}) && (3.50)\\
& +d_{\beta}\overline{\Pi}^{a}\nabla_{a}P^{\beta\overline{\alpha}}+d_{\beta}\overline{\Pi}^{\overline{\gamma}}\nabla_{\overline{\gamma}}P^{\beta\overline{\alpha}}+\lambda^{\beta}\omega_{\gamma}\widetilde{d}_{\overline{\delta}}(C_{\beta}{}^{\rho\overline{\delta}}C_{\rho}{}^{\gamma\overline{\alpha}}-C_{\rho}{}^{\gamma\overline{\delta}}C_{\beta}{}^{\rho\overline{\alpha}}+P^{\rho\overline{\delta}}\nabla_{\rho}C_{\beta}{}^{\gamma\overline{\alpha}}\\
& +P^{\delta\overline{\alpha}}(\nabla_{\delta}C_{\beta}{}^{\gamma\overline{\delta}}-P^{\epsilon\overline{\delta}}R_{\epsilon\delta\beta}{}^{\gamma}))+\lambda^{\beta}\omega_{\gamma}\widetilde{\lambda}^{\overline{\delta}}\widetilde{\omega}_{\overline{\rho}}(S_{\beta\overline{\delta}}{}^{\sigma\overline{\rho}}C_{\sigma}{}^{\gamma\overline{\alpha}}-S_{\sigma\overline{\delta}}{}^{\gamma\overline{\rho}}C_{\beta}{}^{\sigma\overline{\alpha}}\\
& +\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}\nabla_{\sigma}C_{\beta}{}^{\gamma\overline{\alpha}}+P^{\alpha\overline{\epsilon}}(\nabla_{\overline{\epsilon}}S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}}+C_{\beta}{}^{\gamma\overline{\gamma}}\widetilde{R}_{\overline{\gamma}\overline{\epsilon}\overline{\delta}}{}^{\overline{\rho}}+\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}R_{\sigma\overline{\epsilon}\beta}{}^{\gamma}))\\
& -\lambda^{\beta}\omega_{\gamma}\overline{\Pi}^{a}(\nabla_{a}C_{\beta}{}^{\gamma\overline{\alpha}}-R_{a\epsilon\beta}{}^{\gamma}P^{\epsilon\overline{\alpha}})-\lambda^{\beta}\omega_{\gamma}\overline{\Pi}^{\overline{\delta}}S_{\beta\overline{\delta}}{}^{\gamma\overline{\alpha}}+\Pi^{a}\overline{\Pi}^{b}T_{ab}{}^{\overline{\alpha}}-\Pi^{\beta}\overline{\Pi}^{a}T_{a\beta}{}^{\overline{\alpha}}\\
& -\widetilde{d}_{\overline{\beta}}\Pi^{a}P^{\gamma\overline{\beta}}T_{a\gamma}{}^{\overline{\alpha}}-\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^{a}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}T_{a\delta}{}^{\overline{\alpha}}.
\end{aligned}
$$

3.4.1 Beta functions

Now we can obtain the equations for the background fields implied by the vanishing of the beta functions. These are the background dependent expressions for the conformal weights $(1,1)$ independent couplings in (3.46). That is, all the independent combinations formed from the products between $(\Pi^{a},\Pi^{\alpha},d_{\alpha},\lambda^{\alpha}\omega_{\beta})$ and $(\overline{\Pi}^{a},\overline{\Pi}^{\overline{\alpha}},\widetilde{d}_{\overline{\alpha}},\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}})$ because $\Pi^{\overline{\alpha}}$ and $\overline{\Pi}^{\alpha}$ are determined from the equations of motion (3.15). Let us first concentrate on the beta functions coming from the couplings to $\Pi^{A}\overline{\Pi}^{B}$, $d_{\alpha}\overline{\Pi}^{B}$ and $\Pi^{A}\widetilde{d}_{\overline{\beta}}$ fields. After using the results for the expansion (3.30)-(3.38) and the equations (3.49)-(3.50) in (3.46), the couplings $\Pi^{\alpha}\overline{\Pi}^{\overline{\beta}}$, $\Pi^{\alpha}\overline{\Pi}^{b}$, $\Pi^{a}\overline{\Pi}^{\overline{\beta}}$ and $\Pi^{a}\overline{\Pi}^{b}$ lead respectively to a first set of equations

$$
T_{c\overline{\beta}}{}^{\delta}T_{\delta\alpha}{}^{c}-T_{c\alpha}{}^{\overline{\delta}}T_{\overline{\delta}\overline{\beta}}{}^{c}+4\nabla_{\alpha}\nabla_{\overline{\beta}}\Phi=0, \tag{3.51}
$$

$$
\nabla^{d}T_{\alpha db}+R_{\alpha deb}\eta^{de}+T_{bc}{}^{\delta}T_{\delta\alpha}{}^{c}+4\nabla_{b}\nabla_{\alpha}\Phi=0, \tag{3.52}
$$

$$
R_{\overline{\beta}dea}\eta^{de}+T_{ac}{}^{\overline{\delta}}T_{\overline{\delta}\overline{\beta}}{}^{c}-T_{c\overline{\beta}}{}^{\delta}T_{\delta a}{}^{c}+4\nabla_{a}\nabla_{\overline{\beta}}\Phi=0, \tag{3.53}
$$

$$
\eta^{cd}(R_{acdb}+R_{bcda})-\nabla^{c}T_{abc}+T_{c(a}{}^{\alpha}T_{b)\alpha}{}^{c}+8T_{a\alpha}{}^{\overline{\beta}}T_{b\overline{\beta}}{}^{\alpha}+4T_{ab}{}^{c}\nabla_{c}\Phi \tag{3.54}
$$

$$+4T_{ab}{}^{\overline{\alpha}}\nabla_{\overline{\alpha}}\Phi + 4\nabla_a\nabla_b\Phi = 0.$$

We wrote them by increasing their dimensions, that is, if X^a has dimension -1 and each θ^α, $\widetilde{\theta}^{\overline{\alpha}}$ have dimension $-\frac{1}{2}$, then the first has dimension 1, the second and third dimension $\frac{3}{2}$ and the fourth dimension 2. The couplings to $d_\alpha\overline{\Pi}^{\overline{\beta}}$, $\Pi^\alpha\widetilde{d}_{\overline{\beta}}$, $d_\alpha\overline{\Pi}^b$ and $\Pi^a\widetilde{d}_{\overline{\beta}}$ lead respectively to a second set of equations

$$\nabla^c T_{c\overline{\beta}}{}^\alpha - 2\nabla_{\overline{\beta}}P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi + 2P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\nabla_{\overline{\beta}}\Phi = 0, \qquad (3.55)$$

$$\nabla^c T_{c\alpha}{}^{\overline{\beta}} + 2\nabla_\alpha P^{\gamma\overline{\beta}}\nabla_\gamma\Phi - 2P^{\gamma\overline{\beta}}\nabla_\gamma\nabla_\alpha\Phi = 0, \qquad (3.56)$$

$$\nabla^c T_{cb}{}^\alpha - T_{cd}{}^\alpha T_b{}^{cd} + (T_{\delta b}{}^c T_{c\overline{\gamma}}{}^\alpha - R_{b\overline{\gamma}\delta}{}^\alpha)P^{\delta\overline{\gamma}} + T_{b\overline{\gamma}}{}^\delta(3\nabla_\delta P^{\alpha\overline{\gamma}} - 2P^{\alpha\overline{\gamma}}\nabla_\delta\Phi) + 2T_{bc}{}^\alpha\nabla^c\Phi$$
$$\qquad\qquad (3.57)$$
$$-2\nabla_b P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi = 0,$$

$$\nabla^c T_{ca}{}^{\overline{\beta}} - 2T_{cd}{}^{\overline{\beta}}T_a{}^{cd} + P^{\gamma\overline{\beta}}T_\alpha{}^{de}T_{ade} + \widetilde{R}_{a\gamma\overline{\delta}}{}^{\overline{\beta}}P^{\gamma\overline{\delta}} - T_{a\gamma}{}^{\overline{\delta}}(3\nabla_{\overline{\delta}}P^{\gamma\overline{\beta}} - 2P^{\gamma\overline{\beta}}\nabla_{\overline{\delta}}\Phi) \quad (3.58)$$
$$+2T_{ac}{}^{\overline{\beta}}\nabla^c\Phi + 2\nabla_a P^{\gamma\overline{\beta}}\nabla_\gamma\Phi = 0.$$

The first two with have dimension 2 and the second two have dimension $\frac{5}{2}$. Now we will prove that these equations are implied by the classical BRST constraints, the Bianchi identities (3.21) and the relations (3.25).

Firstly, it is important to know the expression for the scale curvature in terms of the scale connection. This are found to be

$$R_{\alpha\beta} = \nabla_{(\alpha}\Omega_{\beta)}, \quad R_{\alpha\overline{\beta}} = \nabla_{\overline{\beta}}\Omega_\alpha, \quad R_{\overline{\alpha}\overline{\beta}} = 0,$$
$$R_{ab} = T_{ab}{}^\gamma\Omega_\gamma, \quad R_{a\beta} = \nabla_a\Omega_\beta, \quad R_{a\overline{\beta}} = T_{a\overline{\beta}}{}^\gamma\Omega_\gamma. \qquad (3.59)$$

$$\widetilde{R}_{\overline{\alpha}\overline{\beta}} = \nabla_{(\overline{\alpha}}\widetilde{\Omega}_{\overline{\beta})}, \quad \widetilde{R}_{\alpha\overline{\beta}} = \nabla_\alpha\widetilde{\Omega}_{\overline{\beta}}, \quad \widetilde{R}_{\alpha\beta} = 0,$$
$$\widetilde{R}_{ab} = T_{ab}{}^{\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}, \quad \widetilde{R}_{a\overline{\beta}} = \nabla_a\widetilde{\Omega}_{\overline{\beta}}, \quad \widetilde{R}_{a\beta} = T_{a\beta}{}^{\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}. \qquad (3.60)$$

Secondly, let us write some expressions useful for later use. We note that the Bianchi identity $(\nabla T)_{\alpha ab}{}^c = 0$, using (3.59) can be written as

$$R_{\alpha[ab]c} = \nabla_\alpha T_{abc} - 2(\gamma_{c[a})_\alpha{}^\beta R_{b]\beta} + (\gamma_c)_{\alpha\beta}T_{ab}{}^\beta - T_{\alpha dc}T_{ab}{}^d - T_{\alpha[a}{}^d T_{b]dc}, \qquad (3.61)$$

now, we can use the identity

$$2R_{\alpha abc} = R_{\alpha[ab]c} + R_{\alpha[ca]b} - R_{\alpha[bc]a}, \tag{3.62}$$

and the Bianchi identity $(\nabla H)_{\alpha abc} = 0$ to write (3.61) as

$$R_{\alpha abc} = T_{a[b}{}^{\beta}(\gamma_{c]})_{\beta\alpha} - 2(\gamma_{bc})_{\alpha}{}^{\beta}R_{a\beta}. \tag{3.63}$$

An identical procedure starting with $(\nabla \widetilde{T})_{\overline{\alpha}ab}{}^c = 0$ allows us to find

$$\widetilde{R}_{\overline{\alpha}abc} = T_{a[b}{}^{\overline{\beta}}(\gamma_{c]})_{\overline{\alpha}\overline{\beta}} - 2(\gamma_{bc})_{\overline{\alpha}}{}^{\overline{\beta}}\widetilde{R}_{a\overline{\beta}}. \tag{3.64}$$

Then, replacing (3.63) and (3.64) respectively in $(\nabla T)_{a\alpha\beta}{}^{\beta} = 0$ and $(\nabla T)_{a\overline{\alpha}\overline{\beta}}{}^{\overline{\beta}} = 0$, we find

$$\gamma^b_{\alpha\beta}T_{ba}{}^{\beta} = 8R_{a\alpha}, \quad \gamma^b_{\overline{\alpha}\overline{\beta}}T_{ba}{}^{\overline{\beta}} = 8\widetilde{R}_{a\overline{\alpha}}. \tag{3.65}$$

We have enough information to show that the equations (3.51) , (3.52) and (3.53) are satisfied. From the Bianchi identity $(\nabla T)_{\alpha\beta\overline{\gamma}}{}^{\beta} = 0$ we obtain

$$T_{\alpha\beta}{}^d T_{d\overline{\gamma}}{}^{\beta} = 17R_{\alpha\overline{\gamma}} + \frac{1}{4}R_{\overline{\gamma}\beta cd}(\gamma^{cd})_{\alpha}{}^{\beta}. \tag{3.66}$$

Since we need an expression for $R_{\overline{\gamma}\beta cd}$, we can use $(\nabla T)_{\alpha\overline{\beta}a}{}^b = 0$, finding

$$R_{\overline{\gamma}\beta cd} = 2(\gamma_{cd})_{\beta}{}^{\delta}\nabla_{\overline{\gamma}}\Omega_{\delta} + T_{c\overline{\gamma}}{}^{\epsilon}(\gamma_d)_{\epsilon\beta} + T_{c\beta}{}^{\overline{\epsilon}}(\gamma_d)_{\overline{\epsilon}\overline{\gamma}}. \tag{3.67}$$

Replacing (3.67) in (3.66) ,using the second equation in (3.59) , $\nabla_{\alpha}\Phi = 4\Omega_{\alpha}$ and the constraints coming from holomorphicity-antiholomorphicity of the BRST current $T_{a\beta}{}^{\overline{\gamma}} = -(\gamma_a)_{\beta\delta}P^{\delta\overline{\gamma}}$, $T_{a\overline{\beta}}{}^{\gamma} = (\gamma_a)_{\overline{\beta}\overline{\delta}}P^{\gamma\overline{\delta}}$ we can verify the equation (3.51).

To verify (3.52) and (3.53), we must contract the a and b indices using η^{ab} in (3.63) and (3.64), and use (3.65) together with the relations (3.25).

For deriving the remaining equation of the first set, the coupling to $\Pi^a \overline{\Pi}^b$, it is useful to find an expression for R_{abcd}, which can be found from the Bianchi identity $(\nabla T)_{ab\alpha}{}^{\beta}$

$$R_{abcd} = -\frac{1}{8}(\gamma_{cd})_{\beta}{}^{\alpha}(\nabla_{\alpha}T_{ab}{}^{\beta} - T_{\alpha[a}{}^e T_{b]e}{}^{\beta} - T_{\alpha[a}{}^{\overline{\gamma}}T_{b]\overline{\gamma}}{}^{\beta}), \tag{3.68}$$

from this equation we construct $\eta^{cd}(R_{acdb} + R_{bcda})$:

$$
\begin{aligned}
\eta^{cd}(R_{acdb} + R_{bcda}) &= -\frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}\nabla_{\alpha}T_{ac}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}\nabla_{\alpha}T_{bc}{}^{\beta}] \\
&\quad + \frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}T_{\alpha[a}{}^e T_{c]e}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}T_{\alpha[b}{}^e T_{c]e}{}^{\beta}] \\
&\quad + \frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}T_{\alpha[a}{}^{\overline{\epsilon}}T_{c]\overline{\epsilon}}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}T_{\alpha[b}{}^{\overline{\epsilon}}T_{c]\overline{\epsilon}}{}^{\beta}].
\end{aligned} \tag{3.69}
$$

Let us consider the right hand side of (3.70) line by line. We can use (3.65) , the Bianchi identity $(\nabla R)_{\alpha a \delta \beta}{}^{\gamma}$ to write

$$(\gamma_b)^{\delta \alpha} \nabla_\alpha R_{a\delta} = -2\nabla_a \nabla_b \Phi - 2T_{ab}{}^{\gamma}\nabla_\gamma \Phi - 2(\gamma_b \gamma_{ae})^{\delta \beta}\Omega_\beta R^e{}_\delta - (\gamma_a \gamma_b)_\beta{}^{\delta} P^{\beta \bar{\epsilon}} R_{\bar{\epsilon}\delta}, \quad (3.70)$$

and the beta function with dimension 1 (3.51) to find the following expression for the first line in the right hand side of (3.70)

$$-4\nabla_b \nabla_a \Phi + 2T_{ab}{}^{C}\nabla_C \Phi - 4\eta_{ab}(\gamma^e)^{\delta \beta}\Omega_\beta R_{e\delta} + 4(\gamma_b)^{\delta \beta}\Omega_\beta R_{a\delta} + 4(\gamma_a)^{\delta \beta}\Omega_\beta R_{b\delta} + \frac{1}{4}\eta_{ab}\eta^{cd}T_{c\beta}{}^{\bar{\delta}}T_{d\bar{\delta}}^{\beta}. \quad (3.71)$$

Finding an expression for the second line is a matter of gamma matrices algebra, once we use (3.59) . For this line we find $\frac{1}{4}\eta_{ab}T_{\beta cd}T^{cd\beta} - \frac{3}{4}T_{c(a}{}^{\beta}T_{b)\beta}{}^{c}$. Using $T_{a\beta}{}^{\bar{\gamma}} = -(\gamma_a)_{\beta\delta}P^{\delta\bar{\gamma}}$ and some gamma matrices algebra, it is straightforward to find $T_{\beta(a}{}^{\bar{\gamma}}T_{b)\bar{\gamma}}{}^{b} - \frac{1}{4}\eta_{ab}\eta^{cd}T_{d\beta}{}^{\bar{\gamma}}T_{c\bar{\gamma}}{}^{\beta}$ for the third line. So, adding the results for the three lines and using (3.65) we find

$$\eta^{cd}(R_{acdb} + R_{bcda}) = -4\nabla_b \nabla_a \Phi - T_{c(a}{}^{\beta}T_{b)\beta}{}^{c} + 2T_{ab}{}^{E}\nabla_E \Phi + T_{\beta(a}{}^{\bar{\gamma}}T_{b)\bar{\gamma}}{}^{\beta}, \quad (3.72)$$

which contains some of the terms in (3.54) . It is also needed to use $(\nabla T)_{abc}{}^{c} = 0$ in order to generate the term $\nabla^c T_{abc}$. This Bianchi identity gives

$$\nabla^c T_{abc} - T_{c[a}{}^{e}T_{b]e}{}^{c} - T_{c[a}{}^{\epsilon}T_{b]\epsilon}{}^{c} - \eta^{cd}(R_{acdb} - R_{bcda}) = 0. \quad (3.73)$$

Finding an expression for $\eta^{cd}(R_{acdb} - R_{bcda})$ is not difficult following the description given to compute (3.72) . After we compute it and replace it in (3.73) we find

$$\nabla^c T_{abc} + T_{\beta[a}{}^{\bar{\delta}}T_{b]\bar{\delta}}{}^{\beta} - 2T_{ab}{}^{c}\nabla_c \Phi + 2T_{ab}{}^{\gamma}\nabla_\gamma \Phi - 2T_{ab}{}^{\bar{\gamma}}\nabla_{\bar{\gamma}} \Phi = 0. \quad (3.74)$$

Combining (3.72) and (3.74) gives the desired beta function equation (3.54).

A similar procedure, but with more steps, is performed to prove the equations of the second group. To probe (3.55) one can start by computing $\{\nabla_{\bar{\alpha}}, \nabla_{\bar{\beta}}\}P^{\gamma\bar{\beta}} = -\gamma^c_{\bar{\gamma}\bar{\beta}}\nabla_c P^{\gamma\bar{\beta}} + \widetilde{R}_{\bar{\alpha}\bar{\beta}\bar{\delta}}{}^{\bar{\beta}}P^{\gamma\bar{\delta}}$. Then we split the curvature as a scale curvature plus a Lorentz curvature. For the latter, use $(\nabla\widetilde{T})_{\bar{\alpha}\bar{\beta}c}{}^{d} = 0$ to obtain

$$\widetilde{R}_{\bar{\alpha}\bar{\beta}cd}(\gamma^{cd})_{\bar{\delta}}{}^{\bar{\beta}} = -180\nabla_{\bar{\alpha}}\widetilde{\Omega}_{\bar{\delta}} + (\gamma^{cd})_{\bar{\delta}}{}^{\bar{\beta}}\nabla_{\bar{\beta}}\widetilde{T}_{\bar{\alpha}cd} + 16\widetilde{T}_{\bar{\delta}}{}^{cd}\widetilde{T}_{\bar{\alpha}cd} + (\gamma^{cd}\gamma^e)_{\bar{\delta}\bar{\alpha}}\widetilde{T}_{ecd}, \quad (3.75)$$

so on one hand we will have

$$\{\nabla_{\overline{\alpha}}, \nabla_{\overline{\beta}}\}P^{\gamma\overline{\beta}} \;=\; -\nabla^c T_{c\overline{\alpha}}{}^\gamma + \widetilde{R}_{\overline{\alpha}\overline{\delta}}P^{\gamma\overline{\delta}} - 45\nabla_{\overline{\alpha}}\widetilde{\Omega}_{\overline{\delta}}P^{\gamma\overline{\delta}} + \frac{1}{4}(\gamma^{cd})_{\overline{\delta}}{}^{\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{T}_{\overline{\alpha}cd}P^{\gamma\overline{\delta}}$$

$$-4\widetilde{T}_{\overline{\alpha}cd}\widetilde{T}_{\overline{\delta}}{}^{cd}P^{\gamma\overline{\delta}} + \frac{1}{4}(\gamma^{cd}\gamma^e)_{\overline{\delta}\overline{\alpha}}\widetilde{T}_{ecd}P^{\gamma\overline{\delta}}. \tag{3.76}$$

On the other hand, we can use $\nabla_{\overline{\alpha}}P^{\beta\overline{\gamma}} = \widetilde{C}_{\overline{\alpha}}{}^{\overline{\gamma}\beta}$, $\widetilde{C}^\gamma = -P^{\gamma\overline{\delta}}\widetilde{\Omega}_{\overline{\delta}}$ and $\widetilde{C}_{cd}{}^\gamma = 1/10(\gamma^a)^{\gamma\alpha}\widetilde{R}_{a\alpha cd}$, which come from antiholomorphicity of the BRST current, to write

$$\{\nabla_{\overline{\alpha}}, \nabla_{\overline{\beta}}\}P^{\gamma\overline{\beta}} = -17\nabla_{\overline{\alpha}}P^{\gamma\overline{\delta}}\widetilde{\Omega}_{\overline{\delta}} - 17P^{\gamma\overline{\delta}}\nabla_{\overline{\alpha}}\widetilde{\Omega}_{\overline{\delta}} + \frac{1}{40}(\gamma^a)^{\gamma\alpha}\nabla_{\overline{\beta}}(\widetilde{R}_{a\alpha cd}(\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}). \tag{3.77}$$

Using $(\nabla\widetilde{T})_{\alpha bcd} = 0$ and $(\widetilde{\nabla}H)_{\alpha bcd} = 0$ it is straightforward to find

$$(\gamma^a)^{\gamma\alpha}\widetilde{R}_{a\alpha cd} = 10T_{cd}{}^\gamma - 10P^{\gamma\overline{\epsilon}}\widetilde{T}_{\overline{\epsilon}cd}. \tag{3.78}$$

Since there is a derivative acting on this terms in (3.77) , we make use of $(\nabla\widetilde{T})_{\overline{\beta}cd}{}^\gamma = 0$ to find

$$(\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}\nabla_{\overline{\beta}}T_{cd}{}^\gamma = -18\nabla^d T_{d\overline{\alpha}}{}^\gamma + (\gamma^{cd}\gamma^e)_{\overline{\alpha}\overline{\delta}}\widetilde{T}_{ecd}P^{\gamma\overline{\delta}} + 16\widetilde{T}_{\overline{\alpha}cd}T_{dc}{}^\gamma. \tag{3.79}$$

We can now replace the last two equations in (3.77) and equate it to (3.76) . The identity

$$(\gamma^{ab})_{(\overline{\alpha}}{}^{\overline{\beta}}(\gamma_{ab})_{\overline{\gamma})}{}^{\overline{\delta}} = -10s\delta_{(\overline{\alpha}}{}^{\overline{\beta}}\delta_{\overline{\gamma})}{}^{\overline{\delta}} + 8(\gamma^a)_{\overline{\alpha}\overline{\gamma}}(\gamma_a)^{\overline{\beta}\overline{\delta}}, \tag{3.80}$$

which can be proved using $(\gamma^a)_{(\overline{\alpha}\overline{\beta}}(\gamma_a)_{\overline{\gamma})\overline{\delta}} = 0$, will be of help to find (3.55). A completely analog procedure allows us to arrive to (3.56).

To prove (3.57) we make use of the Bianchi identities $(\nabla R)_{\alpha ab\beta}{}^\gamma = 0$, $(\nabla T)_{c\alpha\beta}{}^\gamma = 0$ and the identity $(\gamma_a)^{\alpha\beta}R_{\alpha\beta\gamma}{}^\delta = -2(\gamma_a)^{\alpha\beta}R_{\gamma\alpha\beta}{}^\delta$, which follows from $(\nabla T)_{\alpha\beta\gamma}{}^\delta = 0$, to arrive to

$$(\gamma)^{\alpha\beta}(\nabla_\alpha R_{ab\beta}{}^\gamma - 2T_{\alpha[a}{}^e R_{b]e\beta}{}^\gamma - T_{\alpha[a}{}^{\overline{\epsilon}}R_{b]\overline{\epsilon}\beta}{}^\gamma) - 8T_b{}^{ac}T_{ac}{}^\gamma + 8\nabla^a T_{ab}{}^\gamma + 2T_{ab}{}^\gamma\nabla^a\Phi$$

$$-\frac{1}{8}(\gamma)^{\alpha\beta}(\gamma^{cd})_\epsilon{}^\gamma R_{\alpha\beta cd}T_{ab}{}^\epsilon + T_{ab}{}^{\overline{\epsilon}}(\gamma^a)^{\alpha\beta}R_{\overline{\epsilon}\alpha\beta}{}^\gamma = 0. \tag{3.81}$$

The last term in this equation is zero as can easily seen using $(\nabla T)_{\overline{\epsilon}\alpha\beta}{}^\gamma = 0$. The first term can be worked out using (3.68) and $(\nabla T)_{a\overline{\epsilon}\beta}{}^\gamma = 0$, the curvature in the first term of the second line can be rewritten using $(\nabla T)_{\alpha\beta a}{}^b = 0$. The use of $(\nabla T)_{cdb}{}^\delta = 0$ will be also needed to generate (3.57). Again, an analog procedure will allow at arrive to (3.4.1).

So far, we concentrated on a specific set of beta functions. The remaining ones can be classified in a third and fourth sets. The third set involves first order derivatives of the curvatures. We present it again as the dimension increases.

At dimension $5/2$ we find respectively from the couplings to $J\overline{\Pi}^{\overline{\beta}}$, $\Pi^\alpha \widetilde{J}$, $N^{ac}\overline{\Pi}^{\overline{\beta}}$ and $\Pi^\alpha \widetilde{N}^{bc}$

$$\nabla^a R_{a\overline{\beta}} + \nabla_{(\overline{\epsilon}} R_{\overline{\beta})\delta} P^{\delta\overline{\epsilon}} + 2(\nabla_{\overline{\beta}} C^{\overline{\alpha}} - R_{\overline{\beta}\gamma} P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi + 2C^{\overline{\alpha}}\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\Phi = 0, \qquad (3.82)$$

$$\nabla^b \widetilde{R}_{b\alpha} - \nabla_{(\delta} \widetilde{R}_{\alpha)\overline{\epsilon}} P^{\delta\overline{\epsilon}} + 2(\nabla_\alpha \widetilde{C}^\beta + \widetilde{R}_{\alpha\overline{\gamma}} P^{\beta\overline{\gamma}})\nabla_\beta\Phi + 2\widetilde{C}^\beta \nabla_\beta \nabla_\alpha \Phi = 0, \qquad (3.83)$$

$$\nabla^d R_{d\overline{\beta}ac} + \nabla_{(\overline{\epsilon}} R_{\overline{\beta}\delta)ac} P^{\delta\overline{\epsilon}} + 2(\nabla_{\overline{\beta}} C_{ac}{}^{\overline{\alpha}} - R_{\overline{\beta}\gamma ac} P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi + 2C_{ac}{}^{\overline{\delta}}\nabla_{\overline{\delta}}\nabla_{\overline{\beta}}\Phi = 0, \quad (3.84)$$

$$\nabla^d \widetilde{R}_{d\alpha bc} - \nabla_{(\delta} \widetilde{R}_{\alpha)\overline{\epsilon}bc} P^{\delta\overline{\epsilon}} + 2(\nabla_\alpha \widetilde{C}_{bc}{}^\gamma + \widetilde{R}_{\alpha\overline{\delta}bc} P^{\gamma\overline{\delta}})\nabla_\gamma\Phi + 2\widetilde{C}_{bc}{}^\gamma \nabla_\gamma \nabla_\alpha \Phi = 0. \quad (3.85)$$

While at dimension 3 we find respectively from the couplings to $J\overline{\Pi}^b$, $\Pi^a \widetilde{J}$, $N^{ac}\overline{\Pi}^b$ and $\Pi^a \widetilde{N}^{bc}$

$$\nabla^a R_{ab} - T_{ba}{}^c R^a{}_c + T_{ba}{}^\gamma R^a{}_\gamma + 3T_{b\overline{\gamma}}{}^\alpha \nabla_\alpha C^{\overline{\gamma}} + 2R_{bc}\nabla^c\Phi + 2R_{b\overline{\alpha}} P^{\gamma\overline{\alpha}}\nabla_\gamma\Phi$$
$$+2(\nabla_b C^{\overline{\alpha}} - R_{b\gamma} P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi + P^{\delta\overline{\epsilon}}(\nabla_{\overline{\epsilon}} R_{\delta b} + T_{b\delta}{}^c R_{\overline{\epsilon}c} + T_{b\overline{\epsilon}}{}^\gamma R_{\delta\gamma}) = 0, \qquad (3.86)$$

$$\nabla^b \widetilde{R}_{ba} + T_{ab}{}^{\overline{\gamma}} \widetilde{R}^b{}_{\overline{\gamma}} + T_{abc} \widetilde{C}^\delta T_\delta{}^{bc} + 3T_{a\gamma}{}^{\overline{\beta}} \nabla_{\overline{\beta}} \widetilde{C}^\gamma + 2\widetilde{R}_{ab}\nabla^b\Phi - 2\widetilde{R}_{a\gamma} P^{\gamma\overline{\beta}}\nabla_{\overline{\beta}}\Phi$$
$$+2(\nabla_a \widetilde{C}^\beta + \widetilde{R}_{a\overline{\gamma}} P^{\beta\overline{\gamma}})\nabla_\beta\Phi - P^{\delta\overline{\epsilon}}(\nabla_\delta \widetilde{R}_{\overline{\epsilon}a} + T_{\delta a}{}^c \widetilde{R}_{\overline{\epsilon}c} + T_{a\delta}{}^{\overline{\gamma}} \widetilde{R}_{\overline{\epsilon}\overline{\gamma}}) = 0, \qquad (3.87)$$

$$\nabla^d R_{dbac} - T_b{}^{de} R_{deac} + T_b{}^{d\epsilon} R_{d\epsilon ac} + 3T_{b\overline{\delta}}{}^\gamma \nabla_\gamma C_{ac}{}^{\overline{\delta}} + 2R_{bdac}\nabla^d\Phi + 2R_{b\overline{\delta}ac} P^{\epsilon\overline{\delta}}\nabla_\epsilon\Phi \quad (3.88)$$
$$+2(\nabla_b C_{ac}{}^{\overline{\delta}} - R_{b\epsilon ac} P^{\epsilon\overline{\delta}})\nabla_{\overline{\delta}}\Phi + 2R_{b\overline{\delta}ea} C_c{}^{e\overline{\delta}} + P^{\delta\overline{\epsilon}}(\nabla_{\overline{\epsilon}} R_{\delta bac} + T_{b\delta}{}^f R_{\overline{\epsilon}fac} + T_{b\overline{\epsilon}}{}^\gamma R_{\delta\gamma ac}) = 0,$$

$$\nabla^d \widetilde{R}_{dabc} + T_a{}^{d\epsilon} \widetilde{R}_{d\overline{\epsilon}bc} + T_{adf} \widetilde{C}_{bc}{}^\epsilon T_\epsilon{}^{df} + 3T_{a\delta}{}^{\overline{\epsilon}} \nabla_{\overline{\epsilon}} \widetilde{C}_{bc}{}^\delta + 2\widetilde{R}_{adbc}\nabla^d\Phi - 2\widetilde{R}_{a\delta bc} P^{\delta\overline{\epsilon}}\nabla_{\overline{\epsilon}}\Phi \quad (3.89)$$
$$+2(\nabla_a \widetilde{C}_{bc}{}^\delta + \widetilde{R}_{a\overline{\epsilon}bc} P^{\delta\overline{\epsilon}})\nabla_\delta\Phi + 2\widetilde{R}_{a\delta eb} \widetilde{C}_c{}^{e\delta} - P^{\delta\overline{\epsilon}}(\nabla_\delta \widetilde{R}_{\overline{\epsilon}abc} + T_{a\delta}{}^f \widetilde{R}_{\overline{\epsilon}fbc} + T_{a\delta}{}^{\overline{\gamma}} \widetilde{R}_{\overline{\epsilon}\overline{\gamma}bc}) = 0.$$

The fourth set involves second order derivatives of the background fields $P^{\alpha\overline{\beta}}$, $C_\alpha{}^{\beta\overline{\gamma}}$, $\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$ and $S_{\alpha\overline{\beta}}{}^{\gamma\overline{\delta}}$. There is an equation at dimension 3, coming from the coupling to $d_\alpha \widetilde{d}_{\overline{\beta}}$

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$$\nabla^2 P^{\alpha\overline{\beta}} - 2P^{\gamma\overline{\delta}}S_{\gamma\overline{\delta}}{}^{\alpha\overline{\beta}} + T_{de}{}^{\alpha}T^{de\overline{\beta}} - 2\nabla_{\overline{\gamma}}P^{\delta\overline{\beta}}\nabla_{\delta}P^{\alpha\overline{\gamma}} - 2\nabla_c P^{\alpha\overline{\beta}}\nabla^c\Phi \qquad (3.90)$$
$$-2(P^{\gamma\overline{\delta}}\nabla_{\overline{\delta}}P^{\alpha\overline{\beta}} + P^{\alpha\overline{\delta}}\nabla_{\overline{\delta}}P^{\gamma\overline{\beta}})\nabla_{\gamma}\Phi + 2(P^{\delta\overline{\gamma}}\nabla_{\delta}P^{\alpha\overline{\beta}} + P^{\delta\overline{\beta}}\nabla_{\delta}P^{\alpha\overline{\gamma}})\nabla_{\overline{\gamma}}\Phi = 0.$$

At dimension 7/2 we find respectively from the couplings to $J\widetilde{d}_{\overline{\beta}}$, $d_{\alpha}\widetilde{J}$, $N^{ac}\widetilde{d}_{\overline{\beta}}$ and $d_{\alpha}\widetilde{N}^{bc}$

$$\nabla^2 C^{\overline{\beta}} - P^{\alpha\overline{\gamma}}\nabla_{[\alpha}\nabla_{\overline{\gamma}]}C^{\overline{\beta}} - T_{ac}{}^{\overline{\beta}}R^{ac} + 2R_{\gamma}{}^a\nabla_a P^{\gamma\overline{\beta}} + 2\nabla_{\overline{\gamma}}P^{\alpha\overline{\beta}}\nabla_{\alpha}C^{\overline{\gamma}} - C^{\overline{\alpha}}\widetilde{R}_{\overline{\alpha}\delta\overline{\epsilon}}{}^{\overline{\beta}}P^{\delta\overline{\epsilon}} \quad (3.91)$$
$$+P^{\alpha\overline{\beta}}(\nabla_c R_{\alpha}{}^c - \nabla_{[\delta}R_{\overline{\gamma}]\alpha}P^{\delta\overline{\gamma}}) - 2(\nabla_a C^{\overline{\beta}} - P^{\gamma\overline{\beta}}R_{a\gamma})\nabla^a\Phi - 2(P^{\alpha\overline{\beta}}\nabla_{\alpha}C^{\overline{\gamma}} + P^{\alpha\overline{\gamma}}\nabla_{\alpha}C^{\overline{\beta}}$$
$$+P^{\alpha\overline{\beta}}R_{\alpha\gamma}P^{\gamma\overline{\gamma}})\nabla_{\overline{\gamma}}\Phi + 2(SP^{\alpha\overline{\beta}} + \frac{1}{4}\widetilde{S}_{cd}(\gamma^{cd})_{\overline{\epsilon}}{}^{\overline{\beta}}P^{\alpha\overline{\epsilon}} - C^{\overline{\gamma}}\nabla_{\overline{\gamma}}P^{\alpha\overline{\beta}})\nabla_{\alpha}\Phi = 0,$$

$$\nabla^2 \widetilde{C}^{\alpha} - P^{\beta\overline{\gamma}}\nabla_{[\beta}\nabla_{\overline{\gamma}]}\widetilde{C}^{\alpha} - T_{bc}{}^{\alpha}\widetilde{R}^{bc} - 2\widetilde{R}_{\overline{\gamma}}{}^b\nabla_b P^{\alpha\overline{\gamma}} - 2\nabla_{\overline{\gamma}}\widetilde{C}^{\beta}\nabla_{\beta}P^{\alpha\overline{\gamma}} + \widetilde{C}^{\beta}R_{\beta\overline{\epsilon}\delta}{}^{\alpha}P^{\delta\overline{\epsilon}} \quad (3.92)$$
$$-P^{\alpha\overline{\beta}}(\nabla_c \widetilde{R}_{\overline{\beta}}{}^c + \nabla_{[\delta}\widetilde{R}_{\overline{\gamma}]\overline{\beta}}P^{\delta\overline{\gamma}}) - 2(\nabla_b \widetilde{C}^{\alpha} + P^{\alpha\overline{\gamma}}\widetilde{R}_{b\overline{\gamma}})\nabla^b\Phi + 2(P^{\alpha\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{C}^{\gamma} + P^{\gamma\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{C}^{\alpha}$$
$$+P^{\alpha\overline{\epsilon}}\widetilde{R}_{\overline{\epsilon}\overline{\gamma}}P^{\gamma\overline{\gamma}})\nabla_{\gamma}\Phi - 2(SP^{\alpha\overline{\beta}} + \frac{1}{4}S_{cd}(\gamma^{cd})_{\epsilon}{}^{\alpha}P^{\epsilon\overline{\beta}} - \widetilde{C}^{\gamma}\nabla_{\gamma}P^{\alpha\overline{\beta}})\nabla_{\overline{\beta}}\Phi = 0,$$

$$\nabla^2 C_{ac}{}^{\overline{\beta}} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\nabla_{\overline{\epsilon}]}C_{ac}{}^{\overline{\beta}} - R_{deac}T^{de\overline{\beta}} - 2R_{d\epsilon ac}\nabla^d P^{\epsilon\overline{\beta}} + 2\nabla_{\overline{\delta}}P^{\epsilon\overline{\beta}}\nabla_{\epsilon}C_{ac}{}^{\overline{\delta}} - C_{ac}{}^{\overline{\gamma}}\widetilde{R}_{\overline{\gamma}\delta\overline{\epsilon}}{}^{\overline{\beta}}P^{\delta\overline{\epsilon}}$$
$$-P^{\beta\overline{\beta}}(\nabla^d R_{d\beta ac} - \nabla_{[\delta}R_{\overline{\epsilon}]\beta ac}P^{\delta\overline{\epsilon}} + 2R_{\beta\overline{\delta}ea}C_c{}^{e\overline{\delta}}) + 2\nabla_{\overline{\delta}}C_{ea}{}^{\overline{\beta}}C_c{}^{e\overline{\delta}} - 2(\nabla_d C_{ac}{}^{\overline{\beta}} - P^{\epsilon\overline{\beta}}R_{d\epsilon ac})\nabla^d\Phi$$
$$-2(C_{ac}{}^{\overline{\delta}}\nabla_{\overline{\delta}}P^{\gamma\overline{\beta}} - S_{ac}P^{\gamma\overline{\beta}} - \frac{1}{4}S_{acbd}(\gamma^{bd})_{\overline{\delta}}{}^{\overline{\beta}}P^{\gamma\overline{\delta}})\nabla_{\gamma}\Phi$$
$$-2(P^{\gamma\overline{\beta}}\nabla_{\gamma}C_{ac}{}^{\overline{\delta}} + P^{\gamma\overline{\delta}}\nabla_{\gamma}C_{ac}{}^{\overline{\beta}} - P^{\epsilon\overline{\beta}}R_{\epsilon\gamma ac}P^{\gamma\overline{\delta}})\nabla_{\overline{\delta}}\Phi = 0, \qquad (3.93)$$

$$\nabla^2 \widetilde{C}_{bc}{}^{\alpha} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\nabla_{\overline{\epsilon}]}\widetilde{C}_{bc}{}^{\alpha} - \widetilde{R}_{debc}T^{de\alpha} + 2\widetilde{R}_{d\overline{\epsilon}bc}\nabla^d P^{\alpha\overline{\epsilon}} - 2\nabla_{\overline{\delta}}\widetilde{C}_{bc}{}^{\epsilon}\nabla_{\epsilon}P^{\alpha\overline{\delta}} + \widetilde{C}_{bc}{}^{\gamma}R_{\gamma\overline{\epsilon}\delta}{}^{\alpha}P^{\delta\overline{\epsilon}}$$
$$-P^{\alpha\overline{\beta}}(\nabla^d \widetilde{R}_{d\overline{\beta}bc} - \nabla_{[\delta}\widetilde{R}_{\overline{\epsilon}]\overline{\beta}bc}P^{\delta\overline{\epsilon}} + 2\widetilde{R}_{\overline{\beta}\delta eb}\widetilde{C}_c{}^{e\delta}) + 2\nabla_{\delta}\widetilde{C}_{eb}{}^{\alpha}\widetilde{C}_c{}^{e\delta} - 2(\nabla_d \widetilde{C}_{bc}{}^{\alpha} + P^{\epsilon\overline{\beta}}\widetilde{R}_{d\overline{\beta}bc})\nabla^d\Phi$$
$$+2(\widetilde{C}_{bc}{}^{\delta}\nabla_{\delta}P^{\alpha\overline{\gamma}} - \widetilde{S}_{bc}P^{\alpha\overline{\gamma}} - \frac{1}{4}S_{adbc}(\gamma^{ad})_{\delta}{}^{\alpha}P^{\delta\overline{\gamma}})\nabla_{\overline{\gamma}}\Phi$$
$$+2(P^{\alpha\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{C}_{bc}{}^{\delta} + P^{\delta\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{C}_{bc}{}^{\alpha} + P^{\alpha\overline{\epsilon}}\widetilde{R}_{\overline{\epsilon}\overline{\gamma}bc}P^{\delta\overline{\gamma}})\nabla_{\delta}\Phi = 0. \qquad (3.94)$$

Finally, at dimension 4 we find from the couplings to $J\widetilde{J}$, $J\widetilde{N}^{ac}$, $N^{ab}\widetilde{J}$ and $N^{ab}\widetilde{N}^{cd}$ respectively

$$\nabla^2 S - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\nabla_{\overline{\epsilon}]}S - R^{ab}\widetilde{R}_{ab} + 2\widetilde{R}_{a\overline{\beta}}\nabla^a C^{\overline{\beta}} + 2R_{a\beta}\nabla^a\widetilde{C}^{\beta} - 2\nabla_{\overline{\alpha}}\widetilde{C}^{\beta}\nabla_{\beta}C^{\overline{\alpha}}$$

$$-\widetilde{C}^{\beta}(\nabla^a R_{a\beta} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}R_{\bar{\epsilon}]\beta}) - C^{\bar{\beta}}(\nabla^a \widetilde{R}_{a\bar{\beta}} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}\widetilde{R}_{\bar{\epsilon}]\bar{\beta}}) + 2(\widetilde{C}^\alpha R_{b\alpha} + C^{\bar{\alpha}}\widetilde{R}_{b\bar{\alpha}})\nabla^b\Phi$$

$$-2(C^{\bar{\alpha}}\nabla_{\bar{\alpha}}\widetilde{C}^\beta + P^{\beta\bar{\alpha}}(\nabla_{\bar{\alpha}}S + C^{\bar{\gamma}}\widetilde{R}_{\bar{\gamma}\alpha} + \widetilde{C}^\gamma R_{\gamma\bar{\alpha}}))\nabla_\beta\Phi$$

$$-2(\widetilde{C}^\alpha\nabla_\alpha C^{\bar{\beta}} - P^{\alpha\bar{\beta}}(\nabla_\alpha S + C^{\bar{\gamma}}\widetilde{R}_{\bar{\gamma}\alpha} + \widetilde{C}^\gamma R_{\gamma\alpha}))\nabla_{\bar{\beta}}\Phi = 0, \qquad (3.95)$$

$$\nabla^2 \widetilde{S}_{ac} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}\nabla_{\bar{\epsilon}]}\widetilde{S}_{ac} - R^{ed}\widetilde{R}_{edac} + 2\widetilde{R}_{b\bar{\delta}ac}\nabla^b C^{\bar{\delta}} + 2R_{b\delta}\nabla^b\widetilde{C}_{ac}{}^\delta - 2\nabla_{\bar{\beta}}\widetilde{C}_{ac}{}^\delta\nabla_\delta C^{\bar{\beta}}$$

$$-2\nabla_\delta\widetilde{S}_{ba}\widetilde{C}_c{}^{b\delta} - C^{\bar{\beta}}(\nabla^d\widetilde{R}_{d\bar{\beta}ac} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}\widetilde{R}_{\bar{\epsilon}]\bar{\beta}} + 2\widetilde{R}_{\bar{\beta}\delta ea}\widetilde{C}_c{}^{e\delta}) - \widetilde{C}_{ac}{}^\beta(\nabla^d R_{d\beta} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}R_{\bar{\epsilon}]\beta})$$

$$+2(\widetilde{C}_{ac}{}^\beta R_{d\beta} + C^{\bar{\beta}}\widetilde{R}_{d\bar{\beta}ac})\nabla^d\Phi - 2C^{\bar{\delta}}\nabla_{\bar{\delta}}\widetilde{C}_{ac}{}^\gamma\nabla_\gamma\Phi + 4\widetilde{S}_{ab}\widetilde{C}_c{}^{b\gamma}\nabla_\gamma\Phi - 2\widetilde{C}_{ac}{}^\beta\nabla_\beta C^{\bar{\gamma}}\nabla_{\bar{\gamma}}\Phi = 0, \tag{3.96}$$

$$\nabla^2 S_{ab} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}\nabla_{\bar{\epsilon}]}S_{ab} - \widetilde{R}^{cd}R_{cdab} + 2R_{c\delta ab}\nabla^c\widetilde{C}^\delta + 2\widetilde{R}_{c\bar{\delta}}\nabla^c C_{ab}{}^{\bar{\delta}} - 2\nabla_{\bar{\gamma}}\widetilde{C}^\delta\nabla_\delta C_{ab}{}^{\bar{\gamma}}$$

$$-2\nabla_{\bar{\delta}}S_{ca}C_b{}^{c\bar{\delta}} - \widetilde{C}^\gamma(\nabla^d R_{d\gamma ab} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}R_{\bar{\epsilon}]\gamma ab} + 2R_{\gamma\bar{\delta}ea}\widetilde{C}_b{}^{e\bar{\delta}}) - C_{ab}{}^{\bar{\gamma}}(\nabla^d\widetilde{R}_{d\bar{\gamma}} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}R_{\bar{\epsilon}]\bar{\gamma}})$$

$$+2(\widetilde{C}^\gamma R_{d\gamma ab} + C_{ab}{}^{\bar{\gamma}}\widetilde{R}_{d\bar{\gamma}ab})\nabla^d\Phi - 2C_{ab}{}^{\bar{\gamma}}\nabla_{\bar{\gamma}}\widetilde{C}^\delta\nabla_\delta\Phi + 4S_{ac}C_b{}^{c\bar{\delta}}\nabla_{\bar{\delta}}\Phi - 2\widetilde{C}^\gamma\nabla_\gamma C_{ab}{}^{\bar{\delta}}\nabla_{\bar{\delta}}\Phi = 0, \tag{3.97}$$

$$\nabla^2 S_{abcd} - P^{\delta\bar{\epsilon}}\nabla_{[\delta}\nabla_{\bar{\epsilon}]}S_{abcd} - \widetilde{R}^{ef}{}_{cd}R_{efab} + 2\widetilde{R}_{f\bar{\epsilon}cd}\nabla^f C_{ab}{}^{\bar{\epsilon}} + 2R_{feab}\nabla^f\widetilde{C}_{cd}{}^\epsilon$$

$$-2\nabla_{\bar{\epsilon}}\widetilde{C}_{cd}{}^\gamma\nabla_\gamma C_{ab}{}^{\bar{\epsilon}} + 2\nabla_{\bar{\epsilon}}S_{afcd}C_b{}^{f\bar{\epsilon}} + 2\nabla_\epsilon S_{abcd}\widetilde{C}_d{}^{f\epsilon} - C_{ab}{}^{\bar{\epsilon}}(\nabla^e\widetilde{R}_{e\bar{\epsilon}cd} - P^{\delta\bar{\gamma}}\nabla_{[\delta}\widetilde{R}_{\bar{\gamma}]\bar{\epsilon}cd}$$

$$+2\widetilde{R}_{\bar{\epsilon}\delta ec}\widetilde{C}_d{}^{e\delta}) - \widetilde{C}_{cd}{}^\epsilon(\nabla^e R_{e\epsilon ab} - P^{\delta\bar{\gamma}}\nabla_{[\delta}R_{\bar{\gamma}]\epsilon ab}) + 2(\widetilde{C}_{cd}{}^\epsilon R_{e\epsilon ab} + C_{ab}{}^{\bar{\epsilon}}\widetilde{R}_{e\bar{\epsilon}cd})\nabla^e\Phi$$

$$-2C_{ab}{}^{\bar{\gamma}}\nabla_{\bar{\gamma}}\widetilde{C}_{cd}{}^\epsilon\nabla_\epsilon\Phi + 4S_{abcf}\widetilde{C}_d{}^{f\epsilon}\nabla_\epsilon\Phi - 2\widetilde{C}_{cd}{}^\gamma\nabla_\gamma C_{ab}{}^{\bar{\epsilon}}\nabla_{\bar{\epsilon}}\Phi + 4S_{afcd}C_b{}^{f\bar{\epsilon}}\nabla_{eb}\Phi = 0. \tag{3.98}$$

Since the Bianchi identities allow to write the curvature components in terms of the torsion components, we expect that the beta functions of the third set will be implied by the eight beta functions already proven, i.e first and second set. In the same way we expect that the beta functions of the fourth set will also be implied by the first two sets of beta functions since the constraints coming from holomorphicity and antiholomorphicity of the BRST current allows to relate the background fields to some components of the torsion. This is not too hard to check in the case of lower dimension, for example, at dimension $5/2$ consider the beta functions coming from the coupling to $J\overline{\Pi}^{\bar{\beta}}$

$$\nabla^a R_{a\bar{\beta}} + \nabla_{(\bar{\epsilon}}R_{\bar{\beta})\delta}P^{\delta\bar{\epsilon}} + 2(\nabla_{\bar{\beta}}C^{\bar{\alpha}} - R_{\bar{\beta}\gamma}P^{\gamma\bar{\alpha}})\nabla_{\bar{\alpha}}\Phi + 2C^{\bar{\alpha}}\nabla_{\bar{\alpha}}\nabla_{\bar{\beta}}\Phi = 0. \qquad (3.99)$$

By using $R_{a\bar{\beta}} = T_{a\bar{\beta}}{}^\gamma\Omega_\gamma$ and $R_{\bar{\beta}\delta} = \nabla_{\bar{\beta}}\Omega_\delta$, which follow from the definition of the curvature, and $C^{\bar{\beta}} = P^{\alpha\bar{\beta}}\Omega_\alpha$, which follows from the antiholomorphicity constraints, we find that (3.99) can be written as

$$(\nabla^c T_{c\overline{\beta}}{}^\alpha - 2\nabla_{\overline{\beta}} P^{\alpha\overline{\gamma}} \nabla_{\overline{\gamma}}\Phi + 2P^{\alpha\overline{\gamma}} \nabla_{\overline{\gamma}} \nabla_{\overline{\beta}}\Phi)\Omega_\alpha = 0, \qquad (3.100)$$

so, the beta function (3.55) with dimension 2 implies (3.99). Similarly we checked that (3.56) implies (3.83) and that the beta functions with dimension 5/2 (3.57) and (3.4.1) imply respectively the beta functions with dimension 3 (3.86) and (3.87) . This concludes the study of the beta functions for the type II sigma model.

Another application of the superstring sigma model will be presented in the next chapter, based in the heterotic sigma model, in which the quantum consistency of the BRST symmetry will be studied.

Chapter 4

Yang-Mills Chern-Simons Corrections from the Pure Spinor Superstring

The BRST properties play a key role when the superstring is coupled to a generic background. In this chapter it will be shown how these properties can be computed perturbatively in the inverse of the string tension, allowing to find expected Yang-Mills Chern-Simons corrections.

4.1 Lowest Order Constraints in α'

In this section we compute the constraints coming from the nilpotency of the BRST charge and holomorphicity of the BRST current at tree level.

The action which describes the Heterotic Superstring in a curved background can be obtained by adding the massless vertex operators to the flat action and then covariantizing with respec to to the $D = 10$ $N = 1$ super-reparameterization invariance [24], as discussed in chapter 2 . The action is as follows

$$S = \frac{1}{2\pi\alpha'} \int d^2z (\frac{1}{2}\Pi^a\overline{\Pi}^b\eta_{ab} + \frac{1}{2}\Pi^A\overline{\Pi}^B B_{BA} + d_\alpha\overline{\Pi}^\alpha + \Pi^A\overline{J}^I A_{AI} + d_\alpha\overline{J}^I W_I^\alpha \quad (4.1)$$

$$\lambda^\alpha\omega_\beta\overline{J}^I U_{I\alpha}{}^\beta + \lambda^\alpha\omega_\beta\overline{\Pi}^C \Omega_{C\alpha}{}^\beta) + S_\lambda + S_{\overline{J}} + S_\Phi,$$

where $\Pi^A = \partial Z^M E_M^A(Z)$, $\overline{\Pi}^A = \overline{\partial} Z^M E_M^A(Z)$ and $E_M^A(Z)$ is a supervielbein: $G_{MN}(Z) = E_M^a E_N^b \eta_{ba}$. Z^M denote the coordinates for the $D = 10$ $N = 1$ superspace (X^m, θ^μ) with $m = 0, \ldots, 9$ and $\mu = 1, \ldots, 16$. S_λ and $S_{\overline{J}}$, as before, are the actions for λ and $\overline{J}^I = \frac{1}{2}\mathcal{K}_{\mathcal{AB}}^I\bar{\psi}^{\mathcal{A}}\bar{\psi}^{\mathcal{B}}$ respectively, with $\mathcal{A}, \mathcal{B} = 0, \ldots, 32$. S_Φ is the action for the dilaton coupling to the worldsheet scalar curvature. The nilpotency of the BRST charge is guaranteed in a flat background because of the pure spinor condition. Nevertheless, when the superstring is coupled to the curved background, the background fields must be constrained in order to maintain this nilpotency [24]

[57] . We can find these constrains by performing a tree level computation. To set that, we perform a background field expansion [20] by splitting every worldsheet field into a classical and quantum part, where the classical part is assumed to satisfy the classical equation of motion and the quantum part will allow to find propagators and form loops. Specifically, we will use the following notation for the splitting

$$Z^M = X_0^M + Y^M, \ d_\alpha = d_{\alpha 0} + \widehat{d}_\alpha, \tag{4.2}$$

$$\lambda^\alpha = \lambda_0^\alpha + \widehat{\lambda}^\alpha, \ \omega_\alpha = \omega_{\alpha 0} + \widehat{\omega}_\alpha, \ \bar{\psi}^{\mathcal{A}} = \bar{\psi}_0^{\mathcal{A}} + \widehat{\bar{\psi}}^{\mathcal{A}}.$$

So the expansion for the term $\frac{1}{2\pi\alpha'}\int d^2 z \frac{1}{2}\partial Z^M \overline{\partial} Z^N G_{NM}$ in (4.1) in second order of the quantum field is

$$\frac{1}{2\pi\alpha'}\int d^2 z (\frac{1}{2}\partial Y^a \overline{\partial} Y^b \eta_{ab} - \frac{1}{2}\partial Y^a Y^B \overline{\Pi}^C \widetilde{T}_{CB}{}^a - \frac{1}{2}\overline{\partial} Y^a Y^B \Pi^C \widetilde{T}_{CB}{}^a + \frac{1}{4}\partial Y^B Y^C \overline{\Pi}^a \widetilde{T}_{CB}{}^a$$

$$\tag{4.3}$$

$$+\frac{1}{4}\overline{\partial} Y^B Y^C \Pi^a \widetilde{T}_{CB}{}^a + \frac{1}{2} Y^B Y^C \Pi^D \widetilde{T}_{DC}{}^a \overline{\Pi}^E \widetilde{T}_{EB}{}^a - \frac{1}{4} Y^B Y^C \Pi^{(a} \overline{\Pi}^{D)} \widetilde{T}_{DCB}{}^a),$$

where $\widetilde{T}$ is the part of the torsion which only contains derivatives of the vielbein: $\widetilde{T}_{MN}{}^A = \partial_{[M} E_{N]}{}^A$ and $\widetilde{T}_{DCB}{}^A = -\widetilde{T}_{DC}{}^E \widetilde{T}_{EB}{}^A + (-)^{CD}\nabla_C \widetilde{T}_{DB}{}^A$. Note that $\widetilde{T}$ in this chapter is not related to the one used in the last chapter. Repeated bosonic indices in (4.3) are assumed to be contracted with the Minkowski metric. On the other hand, the expansion for $\frac{1}{2\pi\alpha'}\int d^2 z d_\alpha \overline{\partial} Z^M E_M{}^\alpha$ is

$$\frac{1}{2\pi\alpha'}\int d^2 z (\widehat{d}_\alpha \overline{\partial} Y^\alpha - \widehat{d}_\alpha Y^B \overline{\Pi}^C \widetilde{T}_{CB}{}^\alpha + \frac{1}{2}(d_{\alpha 0} + \widehat{d}_\alpha)\overline{\partial} Y^B Y^C \widetilde{T}_{CB}{}^\alpha \tag{4.4}$$

$$-\frac{1}{2}(d_{\alpha 0} + \widehat{d}_\alpha) Y^B \overline{\Pi}^D Y^C (\partial_C \widetilde{T}_{DB}{}^\alpha + \widetilde{T}_{CD}{}^E \widetilde{T}_{EB}{}^\alpha) + \frac{1}{2}\widehat{d}_\alpha \overline{\Pi}^D Y^M Y^N \partial_N E_M{}^B \widetilde{T}_{BD}{}^\alpha)$$

In the subsequent, we will drop off the 0 subindex. From the first term in the last two expressions we can read the propagators

$$Y^a(x,\bar{x})Y^b(z,\bar{z}) \to -\alpha'\eta^{ab}\log|x-z|^2, \ \widehat{d}_\alpha(x)Y^\beta(z) \to \frac{\alpha'\delta_\alpha{}^\beta}{x-z}. \tag{4.5}$$

4.1.1 Nilpotency at tree level

The propagators (4.5) allow to compute the conditions for the nilpotency of Q_{BRST} perturbatively in α'. In fact, we can easily compute a tree level diagram using the second propagator and the fifth term in (4.3) expanding e^{-S} in a series power, giving as a result

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z) = \frac{1}{2}\alpha'\frac{1}{w-z}\lambda^\alpha\lambda^\beta\Pi^c T_{\beta\alpha}{}^c(z). \tag{4.6}$$

The expansions for the remaining terms in the expansion of the action (4.1) are written in the appendix. Initially we are interested in computing the tree level diagrams coming from terms in the expansions with $\overline{\partial}Y^A Y^B$, since they will give rise to the same kind of poles as in (4.6). So, the contributions to the pole $(w-z)^{-1}$ will be

$$\frac{1}{2}\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta\Pi^c(T_{\beta\alpha}{}^c + H^c{}_{\beta\alpha})(z) + \frac{1}{2}\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta\Pi^\gamma H_{\gamma\beta\alpha} \tag{4.7}$$

$$+\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta d_\gamma T_{\beta\alpha}{}^\gamma(z) + \frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta\lambda^\gamma\omega_\delta R_{\beta\alpha\gamma}{}^\delta(z).$$

In our notation, the Torsion superfield $T_{\beta\alpha}{}^\gamma$ is given by

$$T_{\beta\alpha}{}^\gamma = \widetilde{T}_{\beta\alpha}{}^\gamma - \Omega_{\beta\alpha}{}^\gamma - \Omega_{\alpha\beta}{}^\gamma, \tag{4.8}$$

while the curvature superfield is given by

$$R_{\alpha\beta\gamma}{}^\delta = D_\alpha\Omega_{\beta\gamma}{}^\delta + D_\beta\Omega_{\alpha\gamma}{}^\delta + \Omega_{\alpha\gamma}{}^\epsilon\Omega_{\beta\epsilon}{}^\delta + \Omega_{\beta\gamma}{}^\epsilon\Omega_{\alpha\epsilon}{}^\delta + \widetilde{T}_{\alpha\beta}{}^E\Omega_{E\gamma}{}^\delta, \tag{4.9}$$

where D_α is the supersymmetric derivative. There also other possible tree level contractions of $\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)$ with terms including $\partial Y^A Y^B$ which will lead to

$$-\frac{1}{2}\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^c(T_{\beta\alpha}{}^c - H^c{}_{\alpha\beta})(z) + \frac{1}{2}\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^\gamma H_{\gamma\alpha\beta}(z) \tag{4.10}$$

$$-\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{J}^I F_{\alpha\beta I}.$$

In our notation the field-strength superfield is given by

$$F_{\alpha\beta I} = D_\alpha A_{\beta I} + D_\beta A_{\alpha I} + f_I{}^{JK}A_{\alpha J}A_{\beta K} + \widetilde{T}_{\alpha\beta}{}^C A_{CI}. \tag{4.11}$$

To compute the tree-level diagrams that give rise to the above result, we need to compute the integral

$$\int d^2x\frac{1}{(w-x)(x-z)^2} = -\int d^2x\overline{\partial}_x\frac{(\bar{x}-\bar{w})}{x-w}\frac{1}{(x-z)^2} = 2\pi\frac{\bar{w}-\bar{z}}{(w-z)^2} \tag{4.12}$$

From (4.7) and (4.10) we deduce that the conditions for the nilpotency of Q_{BRST} at the lowest order in α' are

$$\lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^C = 0, \quad \lambda^\alpha\lambda^\beta H_{C\alpha\beta} = 0, \quad \lambda^\alpha\lambda^\beta F_{\alpha\beta I} = 0, \quad \lambda^\alpha\lambda^\beta\lambda^\gamma\omega_\delta R_{\beta\alpha\gamma}{}^\delta = 0. \tag{4.13}$$

These are the same set of constraints found in [24] and [57] .

4.1.2 Holomorphicity at tree level

To compute the conditions for holomorphicity of the BRST current $\overline{\partial}j = \overline{\partial}(\lambda^\alpha d_\alpha) = 0$, we must know the expansion up to first order in Y^α of the sigma model action. This expansion for the term $\frac{1}{2\pi\alpha'}\int d^2z \frac{1}{2}\partial Z^M \overline{\partial} Z^N G_{NM}$ is

$$\frac{1}{4\pi\alpha'}\int d^2[\Pi^a\overline{\partial}Y^b\eta_{ab} + \overline{\Pi}^a\partial Y^b\eta_{ab} + \Pi^b\overline{\Pi}^D Y^C \widetilde{T}_{CD}{}^a\eta_{ab} + \Pi^D\overline{\Pi}^a Y^C \widetilde{T}_{CD}{}^b\eta_{ab}]. \quad (4.14)$$

The conditions for holomorphicity will appear as conditions for vanishing to the independent couplings $\Pi^a\overline{\Pi}^b$, $\Pi^\alpha\overline{\Pi}^b$ and so on. For example, forming a tree level diagram contracting $\overline{\partial}d_\alpha$ in $\overline{\partial}j$ with the third term in (4.14) , we obtain $\frac{1}{2}\lambda^\alpha\Pi^b\overline{\Pi}^C \widetilde{T}_{C\alpha}{}^d\eta_{bd}$. Following this procedure with all the terms in the expansion written in the appendix up to order Y, we arrive at

$$\frac{1}{2}\lambda^\alpha[-\Pi^b\overline{\Pi}^c(T_{\alpha b}{}^d\eta_{dc} + T_{\alpha c}{}^d\eta_{bd} + H_{cb\alpha}) + \Pi^\beta\overline{\Pi}^c(T_{\beta\alpha c} - H_{\beta\alpha c}) + \Pi^b\overline{\Pi}^\gamma(T_{\gamma\alpha b} + H_{\gamma\alpha b}) \quad (4.15)$$

$$-\Pi^\beta\overline{\Pi}^\gamma H_{\gamma\beta\alpha} - 2d_\beta\overline{\Pi}^c T_{c\alpha}{}^\beta - 2d_\beta\overline{\Pi}^\gamma T_{\gamma\alpha}{}^\beta + 2\Pi^b\overline{J}^I F_{b\alpha I} + 2\Pi^\beta\overline{J}^I F_{\beta\alpha I} + 2\lambda^\beta\omega_\gamma\overline{\Pi}^d R_{d\alpha\beta}{}^\gamma$$

$$-2d_\beta\overline{J}^I(D_\alpha W_I^\beta - W_J^\beta A_{\alpha K}f_I{}^{JK} - U_{I\alpha}{}^\beta) + 2\lambda^\beta\omega_\gamma\overline{J}^I(\Omega_{\alpha\delta}{}^\gamma U_{I\beta}{}^\delta - \Omega_{\alpha\beta}{}^\delta U_{I\delta}{}^\gamma + U_{J\beta}{}^\gamma A_{\alpha K}f_I^{JK}$$

$$-W_I^\delta R_{\delta\alpha\beta}{}^\gamma - \partial_\alpha U_{I\beta}{}^\gamma)] = 0.$$

Since $\overline{\Pi}^\alpha$ is related to $\overline{J}^I$ through $\overline{\Pi}^\alpha = -\overline{J}^I W_I^\alpha$ by using the equation of motion for the worldsheet field d_α in (4.1) , we arrive at the following set of constraints for the holomorphicity of the BRST current at the lowest order in α'

$$T_{\alpha(bc)} = -H_{abc} = T_{\alpha\beta}{}^c - H_{\alpha\beta}{}^c = T_{c\alpha}{}^\beta = 0, \ \lambda^\alpha\lambda^\beta R_{d\alpha\beta}{}^\gamma = 0, \ F_{\alpha\beta I} = -\frac{1}{2}W_I^\gamma H_{\gamma\alpha\beta}, \quad (4.16)$$

$$F_{\alpha b I} = -W_I^\gamma T_{\gamma\alpha b}, \ \nabla_\alpha W_I^\beta - T_{\alpha\gamma}{}^\beta W_I^\gamma = U_{I\alpha}{}^\beta, \ \lambda^\alpha\lambda^\beta(\nabla_\alpha U_{I\beta}{}^\gamma + R_{\alpha\delta\beta}{}^\gamma W_I^\delta) = 0.$$

This was the same set of constraints found in [24] and [57] .

4.2 Yang-Mills Chern-Simons Corrections

In this section we will compute α' corrections to the nilpotency constraints (4.13) . In the first subsection we will explain how to compute all of the twenty possible contributions to the nilpotency of the BRST charge. In the second subsection, we will explain how, adding some counter-terms, we can find the Yang-Mills Chern-Simons 3$-$form.

4.2.1 One-loop Corrections to the Constraints

In the expansion for the $\Pi^A \overline{J}^I A_{AI}$ term, the following will play a role in our computation: $\Pi^A Y_0^B \overline{J}_2^I(\partial_B A_{AI} + \widetilde{T}_{BA}{}^C A_{CI})(x)$ and $\partial Y^A \overline{J}_2^I A_{AI}(y)$. Contracting them with $\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)$ we can form a 1-loop diagram

$$(4.17)$$

The dashed lines denote background fields while the continuous lines denote the contractions using the propagators. So one can compute how these terms contribute to the nilpotency of Q_{BRST}. To determine the coefficient for this diagram, note that there is an $1/2$ from the expansion of $exp[-S]$ and there is a factor of 2 coming from the possible ways to put the superfields at x or y. Denoting the integration over the world-sheet fields by $\int[Dwsf]$, we find

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_I = \frac{1}{(2\pi\alpha')^2}\int[Dwsf]\int d^2x d^2 y \lambda^\alpha \widehat{d}_\alpha(w)\lambda^\beta \widehat{d}_\beta(z) \qquad (4.18)$$

$$\Pi_0^E Y^\gamma (D_\gamma A_{EI} + \widetilde{T}_{\gamma E}{}^F A_{FI})(x)\partial Y^\delta A_{\delta J}(y)\overline{J}_2^I(x)\overline{J}_2^J(y)$$

$$= \frac{\alpha'^2}{(2\pi)^2}\lambda^\alpha \lambda^\beta \Pi_0^C A_{\alpha I}(D_\beta A_{CI} + \widetilde{T}_{\beta C}{}^D A_{DI})(z)\int d^2 x d^2 y \frac{1}{(w-x)^2(z-y)}\frac{1}{(\bar{x}-\bar{y})^2}$$
$$(4.19)$$

$$- \frac{\alpha'^2}{(2\pi)^2}\lambda^\alpha \lambda^\beta \Pi_0^C A_{\beta I}(D_\alpha A_{CI} + \widetilde{T}_{\alpha C}{}^D A_{DI})(z)\int d^2 x d^2 y \frac{1}{(w-y)(z-x)^2}\frac{1}{(\bar{x}-\bar{y})^2},$$

where $\overline{J}_2^I(\bar{x})\overline{J}_2^J(\bar{y}) \to \frac{(\alpha')^2\delta^{IJ}}{(\bar{x}-\bar{y})^2}$. The second line in the last equation is obtained from minus the first by interchanging α with β and w with z. So, we will just compute one of the integrals.

$$\int d^2 x d^2 y \frac{1}{(w-x)^2(z-y)(\bar{x}-\bar{y})^2} = \int d^2 x d^2 y \frac{1}{(w-x)^2(z-y)}\overline{\partial}_{\bar{y}}\frac{1}{\bar{x}-\bar{y}} \qquad (4.20)$$

$$= 2\pi\int d^2 x d^2 y \frac{\delta^2(y-z)}{(w-x)^2(\bar{x}-\bar{y})} = 2\pi\int d^2 x \frac{1}{(w-x)^2(\bar{x}-\bar{z})},$$
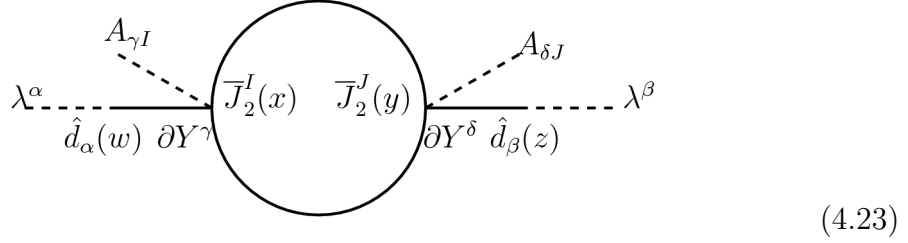
where in the second step we integrated by parts with respect to $\bar{y}$. In the last integral we can integrate by parts with respect to x to obtain

$$\int d^2 x d^2 y \frac{1}{(w-x)^2(z-y)}\frac{1}{(\bar{x}-\bar{y})^2} = -\frac{(2\pi)^2}{w-z}. \qquad (4.21)$$

Then a first contribution to our check of nilpotency will be

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_I = -2\alpha'^2 \frac{\lambda^\alpha \lambda^\beta}{w-z}\Pi_0^C A_{\beta I}(\partial_\alpha A_{CI} + \widetilde{T}_{\alpha C}{}^D A_{DI})(z). \qquad (4.22)$$

A second contribution comes from contracting $\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)$ with $\partial Y^\gamma \overline{J}_2^I A_{\gamma I}(x)\partial Y^\delta \overline{J}_2^J A_{\delta J}(y)$ as shown in the diagram.

$$(4.23)$$

To determine the coefficient of this diagram, note that there is an $1/2$ coming from the Taylor expansion of $exp(-S)$. So we find

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{II} = \frac{\alpha'^2}{2}\frac{\lambda^\alpha \lambda^\beta(z)}{(2\pi)^2}\int d^2x d^2y [\frac{A_{\alpha I}(x)A_{\beta I}(y)}{(w-x)^2(z-y)^2} - \frac{A_{\beta I}(x)A_{\alpha I}(y)}{(w-y)^2(z-x)^2}]\frac{1}{(\bar{x}-\bar{y})^2} \qquad (4.24)$$

The second term in the integrand is obtained from minus the first by interchanging w with z and α with β. The integral we are left to solve is

$$\Gamma = \int d^2x d^2y \frac{A_{\alpha I}(x)A_{\beta I}(y)}{(w-x)^2(z-y)^2(\bar{x}-\bar{y})^2} = -\int d^2x d^2y \frac{\overline{\Pi}^C \partial_C A_{\alpha I}(x)A_{\beta I}(y)}{(\bar{y}-\bar{x})(w-x)^2(z-y)^2} \qquad (4.25)$$

$$+\int d^2x d^2y \frac{A_{\alpha I}(x)A_{\beta I}(y)\partial_x \delta^2(x-w)}{(\bar{y}-\bar{x})(z-y)^2},$$

where we integrated by parts with respect to $\bar{x}$. The first and second integral on the right hand side of (4.25) can be integrated by parts with respect to y and x to obtain

$$\Gamma = 2\pi\int d^2x d^2y \frac{\overline{\Pi}^C \partial_C A_{\alpha I}(x)A_{\beta I}(y)\delta^2(y-x)}{(z-y)(w-x)^2} - 2\pi\int d^2x d^2y \frac{\Pi^C \partial_C A_{\alpha I}(x)A_{\beta I}(y)\delta^2(x-w)}{(\bar{y}-\bar{x})(z-y)^2}. \qquad (4.26)$$

Evaluating the superfields in z, using (4.12) in the first integral and integrating by parts with respect to y in the second, we obtain

$$\Gamma = -(2\pi)^2 \frac{\bar{w}-\bar{z}}{(w-z)^2}\overline{\Pi}^C \partial_C A_{\alpha I}A_{\beta I}(z) - \frac{(2\pi)^2}{w-z}\Pi^C \partial_C A_{\alpha I}A_{\beta I}(z). \qquad (4.27)$$

Then

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{II} = -\alpha'^2 \frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha \lambda^\beta \overline{\Pi}^C \partial_C A_{\alpha I}A_{\beta I}(z) - \frac{\alpha'^2}{w-z}\lambda^\alpha \lambda^\beta \Pi^C \partial_C A_{\alpha I}A_{\beta I}(z) \qquad (4.28)$$

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$$+\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\overline{\partial}\lambda^\alpha\lambda^\beta A_{\alpha I}A_{\beta I}+\frac{\alpha'^2}{w-z}\partial\lambda^\alpha\lambda^\beta A_{\alpha I}A_{\beta I}(z)$$

A third contribution to the nilpotency property comes from contractions of $\Pi_0^A\overline{J}_2^I A_{AI}$, twice $\partial Y^A\overline{J}_2^I A_{AI}$ and $\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)$ giving rise to the diagram

(4.29)

Since we are at order S^3 in the expansion of e^{-S}, there is an $\frac{1}{3!}$ and also a factor of 3 from the possible ways to put the superfields at x, y and u, so there will be a $-1/2$ coefficient in front:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{III}=-\frac{1}{2(2\pi\alpha')^3}\int[Dwsf]\int d^2xd^2yd^2u\lambda^\alpha\widehat{d}_\alpha(w)\lambda^\beta\widehat{d}_\beta(z) \quad (4.30)$$

$$\Pi_0^C\overline{J}_2^I A_{CI}(x)\partial Y^D\overline{J}_2^J A_{DJ}(y)\partial Y^E\overline{J}_2^K A_{EK}(u).$$

$$=-\frac{1}{2(2\pi)^3\alpha'}\lambda^\alpha\lambda^\beta\Pi_0^C A_{CI}A_{\gamma J}A_{\delta K}(z)\int d^2xd^2yd^2u(\frac{\delta_\alpha^{\ \gamma}\delta_\beta^{\ \delta}}{(w-y)^2(z-u)^2} \quad (4.31)$$

$$-\frac{\delta_\alpha^{\ \delta}\delta_\beta^{\ \gamma}}{(w-u)^2(z-y)^2})\overline{J}_2^I(x)\overline{J}_2^J(y)\overline{J}_2^K(u).$$

It is not hard to verify that that

$$\overline{J}_2^I(x)\overline{J}_2^J(y)\overline{J}_2^K(u)=\frac{(\alpha')^3f^{IJK}}{(\bar{x}-\bar{y})(\bar{y}-\bar{u})(\bar{x}-\bar{u})}+\dots, \quad (4.32)$$

where by ... means less singular poles which are not important in this computation. Then the type of integrals we must compute are

$$\Gamma_1=\int d^2xd^2yd^2u\frac{1}{(w-y)^2(z-u)^2(\bar{x}-\bar{y})(\bar{y}-\bar{u})(\bar{x}-\bar{u})}. \quad (4.33)$$

The integral in x gives

$$\int d^2x\frac{1}{(\bar{x}-\bar{y})(\bar{x}-\bar{u})}=\int d^2x\partial_x(\frac{x-y}{\bar{x}-\bar{y}})\frac{1}{\bar{x}-\bar{u}}=-2\pi\frac{y-u}{\bar{y}-\bar{u}}, \quad (4.34)$$

so (4.33) yields

$$\Gamma_1 = -2\pi \int d^2y d^2u \partial_y \left(\frac{1}{w-y}\right) \frac{y-u}{(z-u)^2(\bar{y}-\bar{u})^2}. \tag{4.35}$$

Integrating by parts in y, $\bar{y}$ and then in u we find $\Gamma_1 = (2\pi)^3/(w-z)$. In this way (4.24) gives

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{III} = -(\alpha')^2 \frac{\lambda^\alpha \lambda^\beta}{w-z} f^{IJK} \Pi_0^C A_{CI} A_{\alpha J} A_{\beta K}(z). \tag{4.36}$$

Note that a fourth loop could be formed with $\frac{1}{4}\overline{\partial} Y^\alpha Y^\beta \Pi^c (T_{\beta\alpha}{}^c + H^c{}_{\beta\alpha})$, $\widehat{d}_\alpha \overline{J}_2^I W_I^\alpha$ and $\partial Y^\alpha \overline{J}_2^I A_{\alpha I}$ as shown in the diagram below.

$$\tag{4.37}$$

In this case, we are also at the order S^3, so there is an $\frac{1}{3!}$ which is cancelled by the symmetry factor responsible for the localization of the superfields, either at x, y or u. The $\frac{1}{4}$ coming from the coefficient of the term with Π^c is cancelled by a symmetry factor of the possible ways of contraction:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{IV} = -\frac{\alpha'^2}{(2\pi)^2} \lambda^\alpha \lambda^\beta \Pi^c (T_{\delta\alpha}{}^c + H^c{}_{\delta\alpha}) W_I^\delta A_{\beta I}(z) \tag{4.38}$$

$$\times \int d^2x d^2y d^2u \frac{\delta^2(x-w)}{(z-u)^2(y-x)(\bar{y}-\bar{u})^2}$$

Integrating x we have to solve

$$\int d^2y d^2u \frac{1}{(z-u)^2(y-w)(\bar{y}-\bar{u})^2} = -2\pi \int d^2y d^2u \frac{\delta^2(y-w)}{(\bar{u}-\bar{y})(z-u)^2} = -\frac{(2\pi)^2}{w-z}. \tag{4.39}$$

Then

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{IV} = \frac{\alpha'^2}{w-z} \lambda^\alpha \lambda^\beta \Pi^c (T_{\alpha\delta}{}^c + H^c{}_{\alpha\delta}) W_I^\delta A_{\beta I}(z) \tag{4.40}$$

Considering the same last diagram but with the vertex $\frac{1}{4}\Pi^\gamma H_{\gamma\beta\alpha}$ instead of $\frac{1}{4}\Pi^c (T_{\beta\alpha}{}^c + H_{\beta\alpha}{}^c)$, gives a fifth contribution to the coupling to Π^γ

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_V = \frac{\alpha'^2}{w-z} \lambda^\alpha \lambda^\beta \Pi^\gamma H_{\gamma\alpha\delta} W_I^\delta A_{\beta I}(z) \tag{4.41}$$

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A sixth contribution can be formed with $\frac{1}{4}\Pi^c \overline{\partial} Y^A Y^B (\widetilde{T}_{BA}{}^c + H^c{}_{BA})$ and twice $\partial Y^A \overline{J}_2^I A_{AI}$:

$$\Pi_0^d (\widetilde{T}_{\gamma f}{}^d + H^d{}_{\gamma f}) \tag{4.42}$$

There are 8 possible ways of making the contractions, a 3 factor from the possible ways to put the superfields at x, y or u, an $1/3!$ because we are at S^3 in the expansion, and the factor of $1/4$ of the Π^c term gives a one coefficient:

$$\lambda^\alpha d_\alpha(w) \lambda^\beta d_\beta(z)_{VI} = -\frac{\alpha'^2}{(2\pi)^2} \lambda^\alpha \lambda^\beta \Pi^c (\widetilde{T}_{d\alpha}{}^c + H^c{}_{d\alpha}) A_{dI} A_{\beta I}(z) \times \tag{4.43}$$

$$\int d^2x d^2y d^2u \frac{\delta^2(x - w)}{(y - x)(z - u)^2} \frac{1}{(\bar{y} - \bar{u})^2}.$$

The integral is the same as in (4.38) , so the answer is

$$\lambda^\alpha d_\alpha(w) \lambda^\beta d_\beta(z)_{VI} = \frac{\alpha'^2}{w - z} \lambda^\alpha \lambda^\beta \Pi^c (\widetilde{T}_{d\alpha}{}^c + H^c{}_{d\alpha}) A_{dI} A_{\beta I}(z). \tag{4.44}$$

In the same way, the last diagram but with the vertex $\frac{1}{4}\Pi^\gamma H_{\gamma BA}$ instead of $\frac{1}{4}\Pi^c(T_{BA}{}^c + H_{BA}{}^c)$ leads to a seventh contribution

$$\lambda^\alpha d_\alpha(w) \lambda^\beta d_\beta(z)_{VII} = \frac{\alpha'^2}{w - z} \lambda^\alpha \lambda^\beta \Pi^\gamma H_{\gamma d\alpha} A_{dI} A_{\beta I}(z). \tag{4.45}$$

An eighth contribution can be formed with $-\frac{1}{2}\overline{\partial} Y^a Y^\beta \Pi^C \widetilde{T}_{C\beta}{}^a$ and twice $\partial Y^A \overline{J}_2^I A_{AI}$:

$$\Pi_0^C \widetilde{T}_{C\epsilon}{}^f \tag{4.46}$$

There are 4 possible ways of making the contractions, a 3 factor from the possible ways to put the superfields at x, y or u, an $\frac{1}{3!}$ because we are at S^3 order in the expansion and a factor of $1/2$ of the Π^a coefficient, giving at the end a 1 coefficient:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{VIII} = -\frac{\alpha'^2}{(2\pi)^3}\lambda^\alpha\lambda^\beta\Pi^C\widetilde{T}_{C\alpha}{}^d A_{\beta I}A_{dI}(z)\times \tag{4.47}$$

$$\int d^2x d^2y d^2u \frac{-2\pi\delta^2(u-x)}{(w-x)(z-y)^2}\frac{1}{(\bar{u}-\bar{y})^2}.$$

Integrating in u, the integral we have to solve is

$$\int d^2x d^2y \frac{1}{(w-x)(z-y)^2(\bar{x}-\bar{y})^2} = 2\pi\int d^2x d^2y \frac{\delta^2(x-w)}{(z-y)^2(\bar{y}-\bar{x})} = \frac{(2\pi)^2}{w-z}, \tag{4.48}$$

then

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{VIII} = \frac{\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta\Pi^C\widetilde{T}_{C\alpha}{}^d A_{\beta I}A_{dI}(z). \tag{4.49}$$

Let's consider the couplings to $\overline{\Pi}^A$.

A diagram like (4.37) can be formed with $\frac{1}{4}\overline{\Pi}{}^c \partial Y^A Y^B(\widetilde{T}_{BA}{}^c - H^c{}_{BA})$, $\partial Y^A \overline{J}_2^I A_{AI}$ and $\widehat{d}_\alpha \overline{J}_2^I W_I^\alpha$. There are 4 possible ways of making the contractions, a 6 factor from the possible ways to put the superfields at x, y or u, an $\frac{1}{3!}$ because we are at S^3 order in the expansion and a factor of $1/4$ of the $\overline{\Pi}{}^c$ coefficient, giving at the end a 1 coefficient to this ninth contribution:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{IX} = \frac{\alpha'^2}{(2\pi)^3}\lambda^\alpha\lambda^\beta\overline{\Pi}{}^c(T_{\delta\alpha}{}^c - H^c{}_{\delta\alpha})W_I^\delta A_{\beta I}(z)\times \tag{4.50}$$

$$\int d^2x d^2y d^2u \frac{1}{(w-x)^2(z-u)^2(y-x)(\bar{y}-\bar{u})^2}$$

Integrating $\bar{y}$ by parts, we are left to solve the integral

$$\int d^2x d^2y d^2u \frac{\delta^2(y-x)}{(w-x)^2(z-u)^2(\bar{u}-\bar{y})} = 2\pi\int d^2x \frac{1}{(w-x)(z-x)^2}. \tag{4.51}$$

The right hand side in the last equation is the same as (4.12), so

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{IX} = -\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}{}^c(T_{\delta\alpha}{}^c - H^c{}_{\delta\alpha})W_I^\delta A_{\beta I}(z). \tag{4.52}$$

In the same way, considering vertex $-\frac{1}{4}\overline{\Pi}{}^\gamma H_{\gamma BA}$ instead of $-\frac{1}{4}\overline{\Pi}{}^c(\widetilde{T}_{BA}{}^c - H_{BA}{}^c)$ leads to the tenth contribution

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_X = \alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}{}^\gamma H_{\gamma\delta\alpha}W_I^\delta A_{\beta I}(z) \tag{4.53}$$

An eleventh contribution comes from a diagram like (4.42) which can be formed with $\frac{1}{4}\overline{\Pi}{}^c \partial Y^A Y^B(\widetilde{T}_{BA}{}^c - H^c{}_{BA})$ and twice $\partial Y^A \partial \overline{J}_2^I A_{AI}$. There are 8 possible ways of making the contractions, a 3 factor from the possible ways to put the superfields

at x, y or u, an $\frac{1}{3!}$ because we are at S^3 order in the expansion and a factor of $1/4$ of the $\overline{\Pi}^c$ coefficient, giving at the end a 1 coefficient:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XI} = \frac{\alpha'^2}{(2\pi)^3}\lambda^\alpha\lambda^\beta\overline{\Pi}^c(\widetilde{T}_{d\alpha}{}^c - H^c{}_{d\alpha})A_{dI}A_{\beta I}(z)\times \tag{4.54}$$

$$\int d^2x d^2y d^2u \frac{1}{(w-x)^2(z-u)^2(y-x)(\bar{u}-\bar{y})}.$$

The last integral is the same as the integral in (4.50) , so the result is

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XI} = -\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^c(\widetilde{T}_{d\alpha}{}^c - H^c{}_{d\alpha})A_{dI}A_{\beta I}(z). \tag{4.55}$$

In the same way, a twelfth contribution comes from considering the vertex $-\frac{1}{4}\overline{\Pi}^\gamma H_{\gamma BA}$ instead of the vertex $\frac{1}{4}\overline{\Pi}^c(\widetilde{T}_{BA}{}^c - H_{BA}{}^c)$, leading to

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XII} = \alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^\gamma H_{\gamma d\alpha}A_{dI}A_{\beta I}(z). \tag{4.56}$$

Another diagram like (4.46) can be formed with $-\frac{1}{2}\partial Y^a Y^\beta\overline{\Pi}^C\widetilde{T}_{C\beta}{}^a$, $\partial Y^a\overline{J}_2^I A_{aI}$ and $\partial Y^\alpha\overline{J}_2^I A_{\alpha I}$, giving rise to a thirteenth contribution

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XIII} = -\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\widetilde{T}_{C\alpha}{}^d A_{dI}A_{\beta I}(z). \tag{4.57}$$

A fourteenth contribution and the last for the couplings to $\overline{\Pi}^A$ can be formed with $-\widehat{d}_\alpha Y^B\overline{\Pi}^C\widetilde{T}_{CB}{}^\alpha$ and twice $\partial Y^A\overline{J}A_{AI}$:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XIV} = 2\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C A_{\beta I}\widetilde{T}_{C\alpha}{}^\gamma A_{\gamma I} \tag{4.59}$$

Let's consider the couplings to $\overline{J}_0^I$

A fifteenth contribution to the nilpotency will come from a diagram formed with $\frac{1}{2}\partial Y^A Y^B \overline{J}_0^I(\partial_{[B}A_{A]I} + \widetilde{T}_{BA}{}^C A_{CI})$, $\hat{d}_\alpha \overline{J}_2^I W_I^\alpha$ and $\partial Y^\alpha \overline{J}_2^I A_{\alpha I}$:

$$(4.60)$$

There are 4 possible ways of making the contractions, a 6 factor from the possible ways to put the superfields at x, y or u, an $\frac{1}{3!}$ because we are at the S^3 order in the expansion and a factor of $1/2$ of the $\overline{J}_0^I$ coefficient, giving at the end a 2 factor:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XV} = \frac{2\alpha'^2}{(2\pi)^3}\lambda^\alpha \lambda^\beta \overline{J}_0^I(D_{(\gamma}A_{\alpha)I} + \widetilde{T}_{\gamma\alpha}{}^C A_{CI})W_J^\gamma A_{\beta J}(z)\times \qquad (4.61)$$

$$\int d^2x d^2y d^2u \frac{1}{(w-x)^2(z-u)^2(y-x)(\bar{u}-\bar{y})^2}.$$

The last integral is again the same as in (4.50) , so the result is

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XV} = -2\alpha'^2 \frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha \lambda^\beta \overline{J}_0^I(D_{(\gamma}A_{\alpha)I} + \widetilde{T}_{\gamma\alpha}{}^C A_{CI})W_J^\gamma A_{\beta J}(z). \qquad (4.62)$$

A sixteenth contribution can be formed with $\frac{1}{2}\partial Y^A Y^B \overline{J}_0^I(\partial_{[B}A_{A]I} + \widetilde{T}_{BA}{}^C A_{CI})$ and twice $\partial Y^A \overline{J}_2^I A_{AI}$:

$$(4.63)$$

There are 8 possible ways of making the contractions, a 3 factor from the possible ways to put the superfields at x, y or u, an $\frac{1}{3!}$ because we are at the S^3 order in the expansion and a factor of $1/2$ of the $\overline{J}_0^I$ coefficient, giving at the end a 2 coefficient:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVI} = 2\frac{\alpha'^2}{(2\pi)^3}\lambda^\alpha \lambda^\beta \overline{J}_0^I(\partial_{[c}A_{\alpha]I} + \widetilde{T}_{c\alpha}{}^D A_{DI})A_{cJ}A_{\beta J}(z)\times \qquad (4.64)$$

$$\int d^2x d^2y d^2u \frac{1}{(w-x)^2(z-u)^2(y-x)(\bar{y}-\bar{u})^2},$$

which contains the same integral as before, so the result is

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVI} = -2\alpha'^2 \frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta \overline{J}_0^I(\partial_{[c}A_{\alpha]I}+\widetilde{T}_{c\alpha}{}^D A_{DI})A_{cJ}A_{\beta J}(z). \quad (4.65)$$

Finally, let's consider the couplings to d_α.

A seventeenth contribution can be formed with $\frac{1}{2}d_\alpha\overline{\partial}Y^\beta Y^\gamma \widetilde{T}_{\gamma\beta}{}^\alpha$, $\widehat{d}_\alpha \overline{J}_2^I W_I^\alpha$ and $\partial Y^\alpha \overline{J}_2^I A_{\alpha I}$:

$$(4.66)$$

There are 4 possible ways of making the contractions, a 6 factor from the possible ways to put the superfields at x, y or u, an $\frac{1}{3!}$ because we are at the S^3 order in the expansion and a factor of $1/2$ of the d_α coefficient, giving at the end a 2 coefficient:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVII} = -2\frac{\alpha'^2}{(2\pi)^2}\lambda^\alpha\lambda^\beta d_\gamma\widetilde{T}_{\delta\alpha}{}^\gamma W_I^\delta A_{\beta I}(z) \quad (4.67)$$

$$\times \int d^2x d^2y d^2u \frac{\delta^2(x-w)}{(z-u)^2(y-x)(\bar{y}-\bar{u})^2}$$

Integrating x, the integral we are left to solve is

$$\int d^2y d^2u \frac{1}{(z-u)^2(y-w)(\bar{y}-\bar{u})^2} = -2\pi\int d^2y d^2u \frac{\delta^2(y-w)}{(\bar{u}-\bar{y})(z-u)^2} = -\frac{(2\pi)^2}{w-z}, \quad (4.68)$$

So,

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVII} = \frac{2\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta d_\gamma\widetilde{T}_{\delta\alpha}{}^\gamma W_I^\delta A_{\beta I}(z). \quad (4.69)$$

An eighteenth contribution can be formed with $\frac{1}{2}d_\alpha\overline{\partial}Y^B Y^C \widetilde{T}_{CB}{}^\alpha$ and twice $\partial Y^A \overline{J}_2^I A_{AI}$:
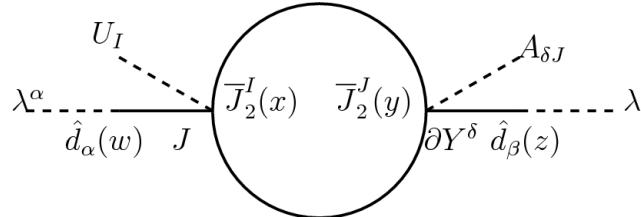

$$(4.70)$$

There are 8 possible ways of making the contractions, a 3 factor from the possible ways to put the superfields at x, y and u, an $\frac{1}{3!}$ because we are at the S^3 order in the expansion and a factor of $1/2$ of the d_α coefficient , giving a 2 coefficient:

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVIII} = 2\frac{\alpha'^2}{(2\pi)^2}\lambda^\alpha\lambda^\beta d_\gamma \widetilde{T}_{c\alpha}{}^\gamma A_{cI}A_{\beta I}(z)\times \tag{4.71}$$

$$\int d^2x d^2y d^2u \frac{\delta^2(x-w)}{(z-u)^2(y-x)(\bar{y}-\bar{u})^2}.$$

This integral is the same as in (4.68) , so the result is

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XVIII} = -\frac{2\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta d_\gamma \widetilde{T}_{c\alpha}{}^\gamma A_{cI}A_{\beta I}(z). \tag{4.72}$$

Because of the pure spinor condition, the action is invariant under $\delta\omega_\alpha = (\Lambda_b\gamma^b\lambda)_\alpha$, so $U_{I\alpha}{}^\beta = U_I\delta_\alpha{}^\beta + \frac{1}{4}U_{Icd}(\gamma^{cd})_\alpha{}^\beta$. We can form a nineteenth one-loop diagram by contracting $J\overline{J}_2^I U_I(x)$ with $\partial Y^\alpha \overline{J}_2^I A_{\alpha I}$:

$$\tag{4.73}$$

giving the contribution

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XIX} = -2\frac{\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta d_\gamma\delta_\alpha{}^\gamma A_{\beta I}U_I \tag{4.74}$$

Similarly, a diagram like (4.73) can be formed contracting $\frac{1}{2}N^{ab}\overline{J}_2^I U_{Iab}(x)$ with $\partial Y^\alpha\overline{J}_2^I A_{\alpha I}$, giving as contribution

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XX} = -\frac{1}{2}\frac{\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta d_\gamma(\gamma^{ef})_\alpha{}^\gamma U_{Ief}A_{\beta I} \tag{4.75}$$

Now, let us summarize our results adding the twenty one-loop contributions to the tree level constraints. Each independent worldsheet coupling will receive corrections, as indicated below:

Corrections to the the coupling to Π^c

$$\frac{1}{2}\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta\Pi^c[(T_{\beta\alpha}{}^c + H^c{}_{\beta\alpha}) - 4\alpha' A_{\beta I}(D_\alpha A_{cI} + \widetilde{T}_{ac}{}^D A_{DI}) + 2\alpha' A_{\beta I}\partial_c A_{\alpha I} \tag{4.76}$$

$$-2\alpha' f^{IJK}A_{cI}A_{\alpha J}A_{\beta K}+2\alpha'(T_{\alpha\delta}{}^c+H^c{}_{\alpha\delta})W_I^\delta A_{\beta I}+2\alpha'(T_{d\alpha}{}^c+T_{c\alpha}{}^e\eta_{ed}+H^c{}_{d\alpha})A_{dI}A_{\beta I}](z).$$

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Corrections to the coupling to $\overline{\Pi}^c$

$$-\frac{1}{2}\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^c[(T_{\beta\alpha}{}^c-H^c{}_{\alpha\beta})-2\alpha'A_{\beta I}\partial_c A_{\alpha I}+2\alpha'(T_{\delta\alpha}{}^c-H^c{}_{\delta\alpha})W_I^\delta A_{\beta I} \quad (4.77)$$

$$+2\alpha'(T_{d\alpha}{}^c+T_{c\alpha}{}^e\eta_{ed}-H^c{}_{d\alpha})A_{dI}A_{\beta I}-4\alpha'A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}](z).$$

Corrections to the coupling to Π^γ

$$\frac{1}{2}\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta\Pi^\gamma[H_{\gamma\beta\alpha}-4\alpha'A_{\beta I}(D_\alpha A_{\gamma I}+\widetilde{T}_{\alpha\gamma}{}^D A_{DI})-2\alpha'A_{\beta I}D_\gamma A_{\alpha I} \quad (4.78)$$

$$-2\alpha'f^{IJK}A_{\gamma I}A_{\alpha J}A_{\beta K}+2\alpha'H_{\gamma\alpha\delta}W_I^\delta A_{\beta I}+2\alpha'(T_{\gamma\alpha d}-H_{\gamma\alpha d})A_{dI}A_{\beta I}](z).$$

Corrections to the coupling to $\overline{\Pi}^\gamma$

$$\frac{1}{2}\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^\gamma[H_{\gamma\alpha\beta}-2\alpha'A_{\beta I}D_\gamma A_{\alpha I}+2\alpha'H_{\gamma\delta\alpha}W_I^\delta A_{\beta I}-2\alpha'(H_{\gamma\alpha}{}^d+T_{\gamma\alpha}{}^d)A_{dI}A_{\beta I}$$

$$(4.79)$$

$$+4\alpha'A_{\beta I}\widetilde{T}_{\gamma\alpha}{}^\delta A_{\delta I}](z).$$

Corrections to the coupling to d_γ

$$\frac{\alpha'}{w-z}\lambda^\alpha\lambda^\beta d_\gamma[T_{\beta\alpha}{}^\gamma+2\alpha'\widetilde{T}_{\delta\alpha}{}^\gamma W_I^\delta A_{\beta I}-2\alpha'\widetilde{T}_{c\alpha}{}^\gamma A_{cI}A_{\beta I}-2\alpha'U_{I\alpha}{}^\gamma A_{\beta I}]. \quad (4.80)$$

Corrections to the coupling to $\overline{J}_0^I$

$$-\alpha'\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{J}^I[F_{\alpha\beta I}+2\alpha'(D_{(\gamma}A_{\alpha)I}+\widetilde{T}_{\gamma\alpha}{}^C A_{CI})W_J^\gamma A_{\beta J} \quad (4.81)$$

$$+2\alpha'(\partial_{[c}A_{\alpha]I}+\widetilde{T}_{c\alpha}{}^D A_{DI})A_{cJ}A_{\beta J}](z).$$

4.2.2 Addition of Counter-terms

Let's now concentrate in finding the Yang-Mills Chern-Simons 3−form by adding appropriate counter-terms. Keeping in mind the lowest order in α' holomorphicity constraints $T_{\alpha bc}+T_{\alpha cb}=0=H_{\alpha bc}$; the conditions for nilpotency at one loop look like
From the coupling to Π^c

$$\lambda^\alpha\lambda^\beta[(T_{\beta\alpha}{}^c+H^c{}_{\beta\alpha})-4\alpha'A_{\beta I}(D_\alpha A_{cI}+\widetilde{T}_{\alpha c}{}^D A_{DI})+2\alpha'A_{\beta I}\partial_c A_{\alpha I} \quad (4.82)$$

$$-2\alpha'f^{IJK}A_{cI}A_{\alpha J}A_{\beta K}+2\alpha'(T_{\alpha\delta}{}^c+H^c{}_{\alpha\delta})W_I^\delta A_{\beta I}](z)=0.$$

From the coupling to $\overline{\Pi}^c$

$$\lambda^\alpha\lambda^\beta[(T_{\beta\alpha}{}^c-H^c{}_{\alpha\beta})-2\alpha'A_{\beta I}\partial_c A_{\alpha I}+2\alpha'(T_{\delta\alpha}{}^c-H^c{}_{\delta\alpha})W_I^\delta A_{\beta I}-4\alpha'A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}](z)=0$$

$$(4.83)$$

Adding (4.82) and (4.83) gives the condition

$$\lambda^\alpha \lambda^\beta [T_{\beta\alpha}{}^c - 2\alpha' A_{\beta I}(\partial_\alpha A_{cI} + \widetilde{T}_{\alpha c}{}^D A_{DI}) - \alpha' f^{IJK} A_{cI} A_{\alpha J} A_{\beta K} + 2\alpha' T_{\alpha\delta}{}^c W_I^\delta A_{\beta I} \quad (4.84)$$

$$-2\alpha' A_{\beta I} \widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}] = 0.$$

Subtracting (4.82) and (4.83) gives the condition

$$\lambda^\alpha \lambda^\beta [H^c{}_{\beta\alpha} - 2\alpha' A_{\beta I}(D_{[\alpha} A_{c]I} + \widetilde{T}_{\alpha c}{}^D A_{DI}) - \alpha' f^{IJK} A_{cI} A_{\alpha J} A_{\beta K} + 2\alpha' H_{\alpha\delta}{}^c W_I^\delta A_{\beta I}$$
$$(4.85)$$
$$+2\alpha' A_{\beta I} \widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}] = 0.$$

Now, suppose that we add a counter-term of the form $\frac{K_1}{2\pi} \int d^2 z \partial Z^M \overline{\partial} Z^N A_{NI} A_{MI}$ to the action, where K_1 is a constant to be determined. This amounts to redefine the space-time metric[59] $G_{MN} \to G_{MN} + 2\alpha' K_1 A_{MI} A_{NI}$. The expansion of this counter-term will contain the terms

$$S_C = \frac{K_1}{2\pi} \int d^2 x [\partial Y^A \overline{\partial} Y^B A_{BI} A_{AI} + \partial Y^A \overline{\Pi}^B A_{BI} Y^C (\partial_C A_{AI} + \frac{1}{2} \widetilde{T}_{CA}{}^D A_{DI}) \quad (4.86)$$

$$+\partial Y^A \overline{\Pi}^B Y^C (\partial_C A_{BI} + \widetilde{T}_{CB}{}^D A_{DI}) A_{AI} + \Pi^A \overline{\partial} Y^B A_{BI} Y^C (\partial_C A_{AI} + \widetilde{T}_{CA}{}^D A_{DI}) +$$

$$\Pi^A \overline{\partial} Y^B Y^C (\partial_C A_{BI} + \frac{1}{2} \widetilde{T}_{CB}{}^D A_{DI}) A_{AI}]$$

which can be used to compute tree level diagrams contracting with $\lambda^\alpha \widehat{d}_\alpha(w) \lambda^\beta \widehat{d}_\beta(z)$. However this diagrams will contribute to the order α'^2, entering at the same foot as the one-loop diagrams. The result of these tree level diagram is

$$-\alpha'^2 K_1 \frac{\overline{w} - \overline{z}}{(w-z)^2} \lambda^\alpha \lambda^\beta \overline{\Pi}^C [A_{CI}(D_{(\alpha} A_{\beta)I} + \widetilde{T}_{\alpha\beta}{}^D A_{DI}) - 2A_{\beta I}(D_\alpha A_{CI} + \widetilde{T}_{\alpha C}{}^E A_{EI})](z)$$
$$(4.87)$$
$$\alpha'^2 K_1 \frac{\lambda^\alpha \lambda^\beta}{w-z} \Pi^C [A_{CI}(D_{(\alpha} A_{\beta)I} + \widetilde{T}_{\alpha\beta}{}^D A_{DI}) - 2A_{\beta I}(D_\alpha A_{CI} + \widetilde{T}_{\alpha C}{}^D A_{DI})](z)$$

$$+2\alpha'^2 K_1 \frac{\overline{w} - \overline{z}}{(w-z)^2} \overline{\partial} \lambda^\alpha \lambda^\beta A_{\alpha I} A_{\beta I}(z) + 2\alpha'^2 \frac{K_1}{w-z} \partial \lambda^\alpha \lambda^\beta A_{\alpha I} A_{\beta I}(z)$$

Then, (4.82) and (4.83) will be modified respectively to

$$\lambda^\alpha \lambda^\beta [(T_{\beta\alpha}{}^c + H^c{}_{\beta\alpha}) - 4\alpha' A_{\beta I}(D_\alpha A_{cI} + \widetilde{T}_{\alpha c}{}^D A_{DI}) + 2\alpha' A_{\beta I} \partial_c A_{\alpha I} \quad (4.88)$$
$$-2\alpha' f^{IJK} A_{cI} A_{\alpha J} A_{\beta K} + 2\alpha'(T_{\alpha\delta}{}^c + H^c{}_{\alpha\delta}) W_I^\delta A_{\beta I} + 2\alpha' K_1 A_{cI}(D_{(\alpha} A_{\beta)I} + \widetilde{T}_{\alpha\beta}{}^D A_{DI})$$
$$-4\alpha' K_1 A_{\beta I}(D_\alpha A_{cI} + \widetilde{T}_{\alpha c}{}^D A_{DI})](z) = 0.$$

$$\lambda^\alpha \lambda^\beta [(T_{\beta\alpha}{}^c - H^c{}_{\alpha\beta}) - 2\alpha' A_{\beta I} \partial_c A_{\alpha I} + 2\alpha'(T_{\delta\alpha}{}^c - H^c{}_{\delta\alpha}) W_I^\delta A_{\beta I} \quad (4.89)$$

$$+2\alpha' K_1 A_{cI}(D_{(\alpha}A_{\beta)I}+\widetilde{T}_{\alpha\beta}{}^D A_{DI})-4\alpha' K_1 A_{\beta I}(D_\alpha A_{cI}+\widetilde{T}_{\alpha c}{}^D A_{DI})-4\alpha' A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}](z) = 0$$

We can add (4.88) with (4.89) to obtain

$$\lambda^\alpha\lambda^\beta[T_{\beta\alpha}{}^c-2\alpha' A_{\beta I}(D_\alpha A_{cI}+\widetilde{T}_{\alpha c}{}^D A_{DI})-\alpha' f^{IJK}A_{cI}A_{\alpha J}A_{\beta K}+2\alpha' T_{\alpha\delta}{}^c W_I^\delta A_{\beta I} \quad (4.90)$$

$$+2\alpha' K_1 A_{cI}(D_{(\alpha}A_{\beta)I}+\widetilde{T}_{\alpha\beta}{}^D A_{DI})-4\alpha' K_1 A_{\beta I}(D_\alpha A_{cI}+\widetilde{T}_{\alpha c}{}^D A_{DI})-2\alpha' A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}] = 0.$$

If $K_1 = -1/2$ and using the constriaint $\lambda^\alpha\lambda^\beta F_{\alpha\beta I} = 0$ we arrive at

$$\lambda^\alpha\lambda^\beta[T_{\beta\alpha}{}^c + 2\alpha' T_{\alpha\delta}{}^c W_I^\delta A_{\beta I} - 2\alpha' A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}] = 0. \quad (4.91)$$

Furthermore, forming a three-level diagram with $\widehat{d}_\alpha Y^\beta \overline{\Pi}^C \widetilde{T}_{C\beta}{}^\alpha$ and $\partial Y^\alpha \overline{\partial} Y^\beta A_{\beta I} A_{\alpha I}$ in (4.86) , with precisely this value for K_1 we can cancel the term proportional to $A_{\beta I}\widetilde{T}_{c\alpha}{}^\gamma A_{\gamma I}$ in (4.91) and (4.85) . Also, with this value for K_1, the counter-terms in the last line of (4.87) will cancel the contributions proportional to $\partial\lambda^\alpha$ and $\overline{\partial}\lambda^\alpha$ in (4.28) .

Note that we can add a second counter-term of the form $\frac{K_2}{2\pi}\int d^2z d_\alpha \overline{\partial} Z^M A_{MI} W_I^\alpha$. This amounts to redefine the supervielbein $E_M{}^\alpha \to E_M{}^\alpha + \alpha' K_2 A_{MI} W_I^\alpha$. After expanding this counter-term, we can form a tree-level diagrams contracting it with $\frac{1}{4}\overline{\partial} Y^\gamma Y^\delta \Pi^c(T_{\delta\gamma}{}^c + H_{\delta\gamma}{}^c)$:

$$(4.92)$$

giving a contribution to the nilpotency

$$\alpha'^2 K_2 \frac{\lambda^\alpha\lambda^\beta}{w - z}\Pi^c(T_{\alpha\gamma}{}^c + H_{\alpha\gamma}{}^c)W_I^\gamma A_{\beta I}(z), \quad (4.93)$$

while contractions with $\frac{1}{4}\partial Y^\gamma Y^\delta \overline{\Pi}^c(T_{\delta\gamma}{}^c - H_{\delta\gamma}{}^c)$ will form the diagram

$$(4.94)$$

which gives the contribution

$$-\alpha'^2 K_2 \frac{\overline{w} - \overline{z}}{(w - z)^2}\lambda^\alpha\lambda^\beta \overline{\Pi}^c(T_{\alpha\gamma}{}^c - H_{\alpha\gamma}{}^c)W_I^\gamma A_{\beta I}. \quad (4.95)$$

It can be easily checked that for $K_2 = -1$, adding (4.93) and (4.95) to (4.82) and (4.83) respectively; then $\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^c$ will not receive α' corrections, i.e. this second counter-term cancels the α' correction in (4.91); while the corrections for $H_{\alpha\beta}{}^c$ are

$$\lambda^\alpha \lambda^\beta [H^c{}_{\beta\alpha} - 2\alpha' A_{\beta I}(D_{[\alpha} A_{c]I} + \widetilde{T}_{\alpha c}{}^D A_{DI}) - \alpha' f^{IJK} A_{cI} A_{\alpha J} A_{\beta K}] = 0. \qquad (4.96)$$

Now, the couplings to Π^γ also receive corrections from the two counter-terms just introduced. Some of these corrections come from the coupling to Π^C in (4.87) when C is γ. Another correction comes from the tree-level diagram

$$\begin{array}{c}
\Pi^\gamma H_{\gamma\delta\phi} \qquad\qquad\qquad\qquad A_{\eta I} W_I^\epsilon \\
\lambda^\alpha \qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda^\beta \\
\hat{d}_\alpha(w) \quad \overline{\partial} Y^\phi Y^\delta(x) \quad \hat{d}_\epsilon \overline{\partial} Y^\eta(y) \quad \hat{d}_\beta(z)
\end{array}$$

$$(4.97)$$

Adding those corrections and using the holomorphicity constraint $F_{\alpha\beta I} = -\frac{1}{2} W_I^\gamma H_{\gamma\alpha\beta}$, we can check that the α' corrections to the coupling to Π^γ are

$$\lambda^\alpha \lambda^\beta [H_{\gamma\beta\alpha} - 2\alpha' A_{\beta I}(D_{(\alpha} A_{\gamma)I} + \widetilde{T}_{\alpha\gamma}{}^D A_{DI}) - \alpha' f^{IJK} A_{\gamma I} A_{\alpha J} A_{\beta K}] = 0. \qquad (4.98)$$

Let's now identify the Chern-Simons form. We can use the lowest order constraints in α' coming from nilpotency condition $\lambda^\alpha \lambda^\beta F_{\alpha\beta I} = 0$ to write (4.96) in the desired form. Since $\lambda^\alpha \lambda^\beta = \lambda^\beta \lambda^\alpha$

$$\lambda^\alpha \lambda^\beta [H^c{}_{\alpha\beta} - \alpha' Tr A_{[\alpha}(D_\beta A_{c]} + \frac{1}{2} \widetilde{T}_{\beta c]}{}^D A_{D]}) - 2\alpha' f^{IJK} A_{cI} A_{\alpha J} A_{\beta K}](z) = 0 \qquad (4.99)$$

Since $2 f^{IJK} A_{cI} A_{\alpha J} A_{\beta K} = \frac{2}{3} Tr A_{[c} A_\alpha A_{\beta]}$ then

$$\lambda^\alpha \lambda^\beta [H^c{}_{\alpha\beta} - \alpha' Tr(A_{[\alpha} D_\beta A_{c]} + \frac{2}{3} A_{[c} A_\alpha A_{\beta]} + \frac{1}{2} A_{[\alpha} \widetilde{T}_{\beta c]}{}^D A_D)](z) = 0, \qquad (4.100)$$

which is the desired form. Similarly, (4.98) can be written as

$$\lambda^\alpha \lambda^\beta [H_{\alpha\beta\gamma} - \alpha' Tr(A_{(\alpha} D_\beta A_{\gamma)} + \frac{2}{3} A_{(\gamma} A_\alpha A_{\beta)} + \frac{1}{2} A_{(\alpha} \widetilde{T}_{\beta\gamma)}{}^D A_D)](z) = 0. \qquad (4.101)$$

Adding a further third counter-term $-\frac{1}{2\pi} \int d^2 z \lambda^\alpha \omega_\beta \overline{\partial} Z^M A_{MI} U_{I\alpha}{}^\beta$, which amounts to redefine $\Omega_{M\alpha}{}^\beta \to \Omega_{M\alpha}{}^\beta - \alpha' A_{MI} U_{I\alpha}{}^\beta$; and thanks also to the other two counter-terms added, can verify that neither $\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^\gamma = 0$ nor $\lambda^\alpha \lambda^\beta F_{\alpha\beta I} = 0$ will receive α' corrections.

There are some similarities between the terms including the gauge connection and the spin connection in the heterotic sigma model action. This suggest that a similar computation would help to find similar Chern-Simons corrections for the gravity side, which will be presented in the next chapter.

Chapter 5

Lorentz Chern-Simons Corrections

In this chapter we consider the Lorentz Chern-Simons type of corrections to the field strength H. To achieve this purpose we consider in the first section the background field expansion of the terms in the action (4.1) that includes the spin connection $\Omega_{M\alpha}{}^\beta$ and compute their α' corrections to the nilpotency of the BRST charge.

5.1 One-loop Correction to the Nilpotency Constraints from Pure Spinors Lorentz Currents

Because the pure spinor condition, (4.1) is invariant under $\delta\omega_\alpha = (\Lambda_b\gamma^b\lambda)_\alpha$. Then

$$\Omega_{M\alpha}{}^\beta = \Omega_M^{(s)}\delta_\alpha{}^\beta + \frac{1}{4}\Omega_{Mab}(\gamma^{ab})_\alpha{}^\beta, \tag{5.1}$$

so the terms including the spin connection can be written

$$\lambda^\alpha\omega_\beta\overline{\Pi}^C\Omega_{C\alpha}{}^\beta = J\overline{\Pi}^C\Omega_C^{(s)} + \frac{1}{2}N^{ab}\overline{\Pi}^C\Omega_{Cab}, \tag{5.2}$$

where $J = \lambda^\alpha\omega_\alpha$ and $N^{ab} = \frac{1}{2}(\lambda\gamma^{ab}\omega)$. Because of the splitting (4.2) J and N^{ab} also splits as

$$J = J_0 + J_1 + J_2, \ \ N^{ab} = N_0^{ab} + N_1^{ab} + N_2^{ab}, \tag{5.3}$$

where the subindex 1 or 2 stands for one or two quantum fields respectively in each definition. Let us now consider the terms in the background field expansion of (5.2) that will allow to form loops. Before that, note that $\lambda^\alpha\omega_\beta\overline{\partial}Z^M\Omega_{M\alpha}{}^\beta$ is analog to $\partial Z^M\overline{J}^I A_{MI}$, then some diagrams of the Yang-Mills Chern-Simons corrections will have a Lorentz analog.

In the expansion for $J\overline{\Pi}^A\Omega_A^{(s)}$ there are terms $J_2\overline{\Pi}^A Y^\beta(D_\beta\Omega_A^{(s)} + \widetilde{T}_{\beta A}{}^D\Omega_D^{(s)})$ and $\overline{\partial}Y^\alpha J_2\Omega_\alpha^{(s)}$ which can be used to form a one-loop diagram like (4.17) contributing to

the nilpotency of the BRST charge:

$$\overline{\Pi}^E(D_\gamma\Omega_E^{(s)} + \widetilde{T}_{\gamma E}{}^C\Omega_C^{(s)})$$
$$\lambda^\alpha \quad \widehat{d}_\alpha(w) \quad Y^\gamma \quad J_2(x) \quad J_2(y) \quad \overline{\partial}Y^\delta \; \widehat{d}_\beta(z) \quad \lambda^\beta \quad \Omega_\delta^{(s)}$$

(5.4)

There is a $1/2$ coming from the expansion at second order or exp(-S), a factor of 2 because the different possibilities of putting the superfields at x or y, a factor of -4, coming from $J_2(x)J_2(y) = -4(x-y)^{-2}$ and a factor of two because of the symmetries of the diagram, giving

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXI} = \frac{1}{(2\pi\alpha')^2}\int[Dwsf]\int d^2x d^2y \lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z) \qquad (5.5)$$

$$\overline{\partial}Y^\gamma\Omega_\gamma^{(s)}(x)\overline{\Pi}_0^E Y^\delta(D_\delta\Omega_E^{(s)} + \widetilde{T}_{\delta E}{}^D\Omega_D^{(s)})(y)J_2(y)J_2(x)$$

$$= -\frac{8\alpha'^2}{(2\pi)}\lambda^\alpha\lambda^\beta\overline{\Pi}_0^E\Omega_\beta^{(s)}(D_\alpha\Omega_E^{(s)} + \widetilde{T}_{\alpha E}{}^D\Omega_D^{(s)})(z)\int d^2x d^2y \frac{\delta^2(x-z)}{(w-y)(x-y)^2}, \qquad (5.6)$$

so

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXI} = -8(\alpha')^2\frac{\overline{w}-\overline{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\Omega_\alpha^{(s)}(\partial_\beta\Omega_C^{(s)} + \widetilde{T}_{\beta C}{}^D\Omega_D^{(s)})(z). \quad (5.7)$$

This result is analog to (4.22).

There is also an one-loop diagram like (4.23) formed contracting $\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)$ with twice $\overline{\partial}Y^\gamma\Omega_\gamma^{(s)}J_2$:

$$\Omega_\gamma^{(s)} \quad \Omega_\delta^{(s)}$$
$$\lambda^\alpha \quad \widehat{d}_\alpha(w) \quad \overline{\partial}Y^\gamma \quad J_2(x) \quad J_2(y) \quad \overline{\partial}Y^\delta \; \widehat{d}_\beta(z) \quad \lambda^\beta$$

(5.8)

which gives a result analog to (4.28)

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXII} = -4\alpha'^2\frac{\overline{w}-\overline{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\partial_C\Omega_\alpha^{(s)}\Omega_\beta^{(s)}(z) - \frac{4\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta\Pi^C\partial_C\Omega_\alpha^{(s)}\Omega_\beta^{(s)}$$

(5.9)

$$-4\alpha'^2\frac{\overline{w}-\overline{z}}{(w-z)^2}\overline{\partial}\lambda^\alpha\lambda^\beta\Omega_\alpha^{(s)}\Omega_\beta^{(s)}(z) - \frac{4\alpha'^2}{w-z}\partial\lambda^\alpha\lambda^\beta\Omega_\alpha^{(s)}\Omega_\beta^{(s)}$$

There is no contribution $\Omega_A^{(s)}\Omega_B^{(s)}\Omega_C^{(s)}$

There is a similar contribution to (5.7) , coming from forming a diagram with $\frac{1}{2}N_2^{ab}\overline{\Pi}^C Y^\delta(D_\delta\Omega_{Cab} + \widetilde{T}_{\delta C}{}^E\Omega_{Eab})$ and $\frac{1}{2}\overline{\partial}Y^C N_2^{ab}\Omega_{Cab}$, which are in the expansion of $\frac{1}{2}N_2^{ab}\overline{\Pi}^C\Omega_{Cab}$:

$$(5.10)$$

giving as result

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXIII} = -3(\alpha')^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\Omega_{\alpha de}(D_\beta\Omega_C{}^{ed} + \widetilde{T}_{\beta C}{}^F\Omega_F{}^{ed})(z).$$

$$(5.11)$$

Also there is a similar contribution to (5.9) , making a diagram contracting $\lambda^\alpha\widehat{d}_\alpha(w)\lambda^\beta\widehat{d}_\beta(z)$ with twice $\frac{1}{2}\overline{\partial}Y^\gamma N_2^{ab}\Omega_{\beta ab}$:

$$(5.12)$$

giving as result

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXIV} = -\frac{3}{2}\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\partial_C\Omega_{\alpha ab}\Omega_\beta{}^{ba}(z) \qquad (5.13)$$

$$-\frac{3}{2}\frac{\alpha'^2}{w-z}\lambda^\alpha\lambda^\beta\Pi^C\partial_C\Omega_{\alpha ab}\Omega_\beta{}^{ba}-\frac{3}{2}\alpha'^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\overline{\partial}\lambda^\alpha\lambda^\beta\Omega_{\alpha ab}\Omega_\beta{}^{ba}(z)-\frac{3}{2}\frac{\alpha'^2}{w-z}\partial\lambda^\alpha\lambda^\beta\Omega_{\alpha ab}\Omega_\beta{}^{ba}.$$

There is a diagram like (4.29) which gives a cubic contribution in Ω, coming from contracting $\frac{1}{2}N_2^{ab}\overline{\Pi}^C\Omega_{Cab}$ and a product of two $\frac{1}{2}\overline{\partial}Y^\alpha N_2^{bc}\Omega_{\alpha bc}$:

$$(5.14)$$

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To determine the coefficient of this diagram, note that there is a $\frac{1}{3!}$ coming from the expansion of $exp(-S)$ at third order, a factor of 3 because the different ways to put the superfields at x, y or u, an $\frac{1}{8}$ coming from the one halves in each of the three terms and a factor of 2 because of the possible ways of contracting, giving

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXV} = -\frac{1}{8(2\pi)^3\alpha'}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\Omega_{Cab}\Omega_{\alpha cd}\Omega_{\beta ef}(z)\times \tag{5.15}$$

$$\int d^2x d^2y d^2u(2\pi)^2\delta^2(x-w)\delta^2(y-z)N_2^{ab}(x)N_2^{cd}(y)N_2^{ef}(u),$$

It is not hard to compute

$$N_2^{ab}(x)N_2^{cd}(y)N_2^{ef}(u) = \frac{-3(\alpha')^3(\eta^{bc}\eta^{a[f}\eta^{e]d}-\eta^{ac}\eta^{b[f}\eta^{e]d}-\eta^{bd}\eta^{a[f}\eta^{e]c}+\eta^{ad}\eta^{b[f}\eta^{e]c})}{(x-y)(y-u)(x-u)}+\ldots, \tag{5.16}$$

where by $\ldots$ is meant less singular terms which are not of importance in this computation. Then

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXV} = \frac{3\alpha'^2}{2\pi}\lambda^\alpha\lambda^\beta\Pi_0^C\Omega_{Cab}\Omega_\alpha{}^{be}\Omega_{\beta e}{}^a(z)\times\int d^2u\frac{1}{(w-z)(z-u)(w-u)}. \tag{5.17}$$

The type of the last integral was already solved and has the form of (4.34), then we arrive to an answer analog to (4.36)

$$\lambda^\alpha d_\alpha(w)\lambda^\beta d_\beta(z)_{XXV} = -3(\alpha')^2\frac{\bar{w}-\bar{z}}{(w-z)^2}\lambda^\alpha\lambda^\beta\overline{\Pi}^C\Omega_{Cde}\Omega_\alpha{}^{ef}\Omega_{\beta f}{}^d(z) \tag{5.18}$$

With the computations of the present section, the α' corrections to the nilpotency condition for the couplings to Π^c and $\overline{\Pi}^c$ are respectively

$$\lambda^\alpha\lambda^\beta[(T_{\beta\alpha}{}^c + H_{\beta\alpha}{}^c) + 8\alpha'\Omega_\beta^{(s)}\partial_c\Omega_\alpha^{(s)} + 3\alpha'\Omega_\beta{}^{ba}\partial_c\Omega_{\alpha ab}] = 0 \tag{5.19}$$

$$\lambda^\alpha\lambda^\beta[(T_{\beta\alpha}{}^c - H_{\beta\alpha}{}^c) + 16\alpha'\Omega_\beta^{(s)}(D_\alpha\Omega_c^{(s)} + \widetilde{T}_{\alpha c}{}^D\Omega_D^{(s)}) - 8\alpha'\Omega_\beta^{(s)}\partial_c\Omega_\alpha^{(s)} \tag{5.20}$$

$$+6\alpha'\Omega_{\beta ab}(D_\alpha\Omega_c{}^{ba} + \widetilde{T}_{\alpha c}{}^D\Omega_D{}^{ba}) - 3\alpha'\Omega_\beta{}^{ba}\partial_c\Omega_{\alpha ab} + 6\alpha'\Omega_{cab}\Omega_\alpha{}^{bd}\Omega_{\beta d}{}^a] = 0,$$

so adding and subtracting (5.19) and (5.20) we obtain respectively

$$\lambda^\alpha\lambda^\beta[T_{\beta\alpha}{}^c + 8\alpha'\Omega_\beta^{(s)}(D_\alpha\Omega_c^{(s)} + \widetilde{T}_{\alpha c}{}^E\Omega_E^{(s)}) + 3\alpha'\Omega_{\beta ab}(D_\alpha\Omega_c{}^{ba} + \widetilde{T}_{\alpha c}{}^E\Omega_E{}^{ba}) \tag{5.21}$$

$$+3\alpha'\Omega_{cab}\Omega_\alpha{}^{bd}\Omega_{\beta d}{}^a] = 0$$

$$\lambda^\alpha\lambda^\beta[H_{\beta\alpha}{}^c - 8\alpha'\Omega_\beta^{(s)}(D_{[\alpha}\Omega_{c]}^{(s)} + \widetilde{T}_{\alpha c}{}^E\Omega_E^{(s)}) - 3\alpha'\Omega_{\beta ab}(D_{[\alpha}\Omega_{c]}{}^{ba} + \widetilde{T}_{\alpha c}{}^E\Omega_E{}^{ba}) \tag{5.22}$$

$$-3\alpha'\Omega_{cab}\Omega_\alpha{}^{bd}\Omega_{\beta d}{}^a] = 0$$

Using the lowest order in α' constraint $\lambda^\alpha\lambda^\beta R_{\beta\alpha cd} = 0$, we can write (5.22) as

$$\lambda^\alpha\lambda^\beta[H_{\beta\alpha}{}^c - 8\alpha'\Omega_\beta^{(s)}(D_{[\alpha}\Omega_{c]}^{(s)} + \widetilde{T}_{\alpha c}{}^E\Omega_E^{(s)}) - \frac{3}{2}\alpha'(\Omega_{\beta a}{}^b(D_{[\alpha}\Omega_{c]b}{}^a + \widetilde{T}_{\alpha c}{}^E\Omega_{Eb}{}^a) \quad (5.23)$$

$$+\Omega_{\alpha a}{}^b(D_{[\beta}\Omega_{c]b}{}^a + \widetilde{T}_{\beta c}{}^E\Omega_{Eb}{}^a) + \Omega_{ca}{}^b(D_{(\alpha}\Omega_{\beta)b}{}^a + \widetilde{T}_{\alpha\beta}{}^E\Omega_{Eb}{}^a) + 4\Omega_{ca}{}^b\Omega_{\alpha b}{}^d\Omega_{\beta d}{}^a)] = 0.$$

To use the same notation as in the gauge case, let's use the same representation as in that case. Let's write $\Omega_{Ab}{}^c = \Omega_{AI}(T^I)_b{}^c$, where $(T^IT^J - T^JT^I)_b{}^c = f^{IJ}{}_K(T^K)_b{}^c$ and $(T^I)_b{}^c(T^J)_c{}^b = 2\delta^{IJ}$. Using this notation (5.23) can be written as

$$\lambda^\alpha\lambda^\beta[H_{\beta\alpha}{}^c - 4\alpha'\Omega_{[\beta}^{(s)}(D_\alpha\Omega_{c]}^{(s)} + \frac{1}{2}\widetilde{T}_{\alpha c}{}^E\Omega_E^{(s)}) - 3\alpha'Tr(\Omega_{[\beta}(D_\alpha\Omega_{c]} + \frac{1}{2}\widetilde{T}_{\alpha c]}{}^E\Omega_E) \quad (5.24)$$

$$+\frac{2}{3}\Omega_{[c}\Omega_\alpha\Omega_{\beta]})] = 0.$$

Which gives the desired form of the Lorentz Chern-Simons.

Summarizing, the Yang-Mills and Lorentz Chern-Simons corrections are

$$\lambda^\alpha\lambda^\beta[H_{\beta\alpha c} - \alpha'Tr(A_{[\alpha}D_\beta A_{c]} + \frac{2}{3}A_{[\alpha}A_\beta A_{c]} + \frac{1}{2}A_{[\alpha}\widetilde{T}_{\beta c]}{}^D A_D) \quad (5.25)$$

$$-3\alpha'Tr(\Omega_{[\alpha}D_{(\beta}\Omega_{c]} + \frac{2}{3}\Omega_{[\alpha}\Omega_\beta\Omega_{c]} + \frac{1}{2}\Omega_{[\alpha}\widetilde{T}_{\beta c]}{}^D\Omega_D) - 4\alpha'\Omega_{[\beta}^{(s)}(D_\alpha\Omega_{c]}^{(s)} + \frac{1}{2}\widetilde{T}_{\alpha c}{}^E\Omega_E^{(s)})] = 0$$

There are further one-loop diagrams that can be formed with terms in the expansion containing three quantum fields. It's computation constitutes work in progress.

Chapter 6

Conclusions

This thesis covered two applications of the non-linear sigma models, namely the computation of equations of motion for the background fields coupled to the bosonic and type II superstring and also the appearance of the Yang-Mills Chern-Simons three-form for the heterotic superstring.

The first application was explained in detail for the bosonic string and for the type II superstring using the pure spinor formalism. Both of them are conformally invariant in a flat space, so when they are coupled to a generic background, which has a direct correspondence with the massless states in each case, the conformal invariance must be checked. The background field method, useful to obtain a covariant expansion was discussed in detail for the bosonic string computations. A version adapted to superspace [20] of the background field method was used to obtain the expansions for the type II and heterotic string. The result of expanding an action using this method allowed to form Feynman diagrams at one loop, contributing to the possible lack of invariance of conformal symmetry at the quantum level. When all those diagrams were computed, giving contributions to the beta functions, it was shown that for the bosonic sigma model, these beta functions can be set consistently to zero. In the introduction it was also presented a spacetime action from which the conditions for conformal invariance can be obtained as equations of motion by a simple variation of this action in space-time. The necessity to use another formalism to make the computations for the superstring is supported because neither the RNS nor the GS sigma model can not be covariantly quantized and at the same time include all the background fields. The pure spinor formalism was briefly discussed in the introduction, and the non-linear sigma model for the heterotic and type II superstring were discussed in more detail. It was explained how the properties of nilpotency of the pure spinor BRST charge and the conservation of its corresponding current allows to find constraints on the background fields at the lowest order

in α': super $N = 1$ $D = 10$ Yang-Mills/supergravity for the heterotic and $N = 2$ $D = 10$ for the type II superstring. For the heterotic string, it was explained by two different methods how to arrive to those constraints: defining canonical momenta and using Poisson brackets, as explained in chapter 2, or by performing a tree level computation as explained in chapter 4.

In the one-loop computation of the beta functions for the type II superstring it was necessary to introduce a scale Λ to regulate the diagrams. By studying the conditions under which the theory does not depend on that scale, a set of equations was computed in chapter 3, corresponding to all the independent couplings to products of worldsheet fields. Because of the background field expansion used, the result of the one-loop computation has super-Poincaré symmetry. With the help of some Bianchi identities and gauge invariances of the sigma model action, some components of the torsion where gauge fixed. Also, the scales connections Ω_α and $\widetilde{\Omega}_{\bar\alpha}$ where related to the derivatives of the dilaton $\nabla_\alpha \Phi$ and $\nabla_{\bar\alpha} \Phi$ respectively. It was verified for the lowest dimension equations of motion that the lowest order α' type II supergravity constraints can set the beta-functions to zero, implying in this way in conformal invariance. This is a straightforward, although non-trivial task, whose level of difficulty increases as one considers equations of motion with higher dimension.

The second application concerns the quantum regime of the BRST symmetry for the heterotic string sigma in the pure spinor formalism. A similar background field expansion as the one used for the pure spinor type II sigma model was included in the appendix, where the gauge and spin connections appear explicitly. One-loop diagrams were formed as a result of considering the product of two BRST charges evaluated in different points. The result of computing these diagrams has poles structure as the two points are approached. This pole structure are of two types: double poles and $(\bar{w} - \bar{z})(w - z)^{-2}$ poles, coupling to independent worldsheet fields. From the set of equations obtained by imposing the vanishing of the poles coefficients, corrections of the order α' are obtained for the classical nilpotency conditions. Chapter 4 was focused in the computations including the gauge fields, which allowed to find the Yang-Mills Chern-Simons three-forms correction of α' order to the field strength of the two-form superpotential, or Kalb-Ramond superfield: $H_{MNP} \rightarrow H_{MNP} - \alpha' \omega_{MNP}$. These corrections are known since the studies of $N = 1$ super Yang-Mills coupled to $N = 1$ supergravity in 10 dimensions [44], [10], [43]. Other interesting redefinition of the fields of α' order were found, such as a redefinition of the metric superfield $G_{MN} \rightarrow G_{MN} - \alpha' Tr A_M A_N$, also known since the preservation of the $N = 1$ supersymmetry at the quantum level

in the RNS sigma model [59]. A field redefinition not known until now was found $E_M{}^\alpha \to E_M{}^\alpha - \alpha' A_{MI} W_I^\alpha$, since this component of the super-vielbein does not appear neither in the RNS nor GS sigma models.

A perspective of the present thesis is to compute the Lorentz Chern-Simons corrections. Partial results were presented in chapter 5, in which similar diagrams to the gauge side were computed. Using the Lorentz connection $\Omega_{Ma}{}^b$ a Lorentz Chern-Simons three-form can be identified, although the role of a Lorentz Chern-Simons three form formed with the scale connection $\Omega^{(s)}$ is not yet understood. The results presented in chapter 5 concerned only computations involving the pure spinors Lorentz currents $N^{ab} = \frac{1}{2}\lambda\gamma^{ab}\omega$ and pure spinor ghost current $J = \lambda^\alpha d_\alpha$. Nevertheless, because of the holomorphicity and nilpotency constraints at the lowest order in α' and using some symmetries of the action, some components of the torsion can be gauged fixed to zero, allowing to write $\widetilde{T}_{c\alpha}{}^\beta = \Omega_{c\alpha}{}^\beta$ and $\widetilde{T}_{\gamma\alpha}{}^\beta = \Omega_{\gamma\alpha}{}^\beta + \Omega_{\alpha\gamma}{}^\beta$. Furthermore, the spin connection can be written in terms of a Lorentz and scale connections. Considering these facts in the background field expansion for the term $\frac{1}{2\pi\alpha'}\int d_\alpha\overline{\partial}Z^M E_M{}^\alpha$, written in equation (4.4) and denoting by $M^{ab} = \frac{1}{2}d_\alpha(\gamma^{ab})^\alpha{}_\beta Y^\beta$, this expansion can be written as follows

$$\frac{1}{2\pi\alpha'}\int d^2z[-dY\overline{\Pi}^d\Omega_d^{(s)}+\frac{1}{2}M^{ab}\overline{\Pi}^d\Omega_{dab}-dY\overline{\partial}Y^\gamma\Omega_\gamma^{(s)}+\frac{1}{2}M^{ab}\overline{\partial}Y^\gamma\Omega_{\gamma ab}-\frac{1}{2}\overline{\partial}d_\alpha Y^\alpha Y^\beta\Omega_\beta^{(s)}$$
$$(6.1)$$
$$+\frac{1}{8}(\overline{\partial}d\gamma^{ab}Y)Y^\beta\Omega_{\beta ab}-\frac{1}{2}dYY^\beta\overline{\Pi}^d\partial_d\Omega_\beta^{(s)}+\frac{1}{4}M^{ab}Y^\beta\overline{\Pi}^d\partial_d\Omega_{\beta ab}-\frac{1}{2}dY\overline{\Pi}^dY^\gamma(\partial_\gamma\Omega_d^{(s)}+\widetilde{T}_{\gamma d}{}^E\Omega_E^{(s)})$$
$$+\frac{1}{4}M^{ab}\overline{\Pi}^dY^\gamma(\partial_\gamma\Omega_{dab}+\widetilde{T}_{\gamma d}{}^E\Omega_{Eab})-\frac{1}{2}d_\alpha Y^\beta\overline{\Pi}^dY^\gamma\widetilde{T}_{\gamma d}{}^\alpha\Omega_\beta^{(s)}-\frac{1}{8}d_\alpha Y^\beta\overline{\Pi}^dY^\gamma\widetilde{T}_{\gamma d}{}^\epsilon\Omega_{\beta ab}(\gamma^{ab})_\epsilon{}^\alpha$$
$$+\frac{(-)^{d(b+p)}}{2}\widehat{d}_\alpha\overline{\Pi}^CY^BY^DE_B{}^PE_D{}^N\partial_NE_P{}^\alpha,]$$

where dY denotes the current $dY = d_\alpha Y^\alpha$ and it satisfy

$$dY(y)dY(z) \to 16\alpha'^2(y-z)^{-2}, \qquad (6.2)$$

while the Lorentz currents M^{ab} have the following OPE

$$M^{ab}(y)M^{cd}(z) \to \alpha'\frac{\eta^{c[b}M^{a]d}(z) - \eta^{d[b}M^{a]c}(z)}{y-z} + 4\alpha'^2\frac{\eta^{a[d}\eta^{c]b}}{(y-z)^2}, \qquad (6.3)$$

which is not surprising since this is the Lorentz current algebra of Siegel approach to the Green-Schwarz superstring. It is expected that including the one-loop diagrams contributing to the nilpotency of the BRST charge, formed with the terms in (6.3), the -3 coefficient in front of the Lorentz Chern-Simons three-form (5.25) will

turn into $+1$ and also the relative coefficient between the Chern-Simons three-forms constructed with the scale connection and with the Lorentz connection can be understood.

Some one-loop diagrams have been computed formed with the terms in the expansion (6.1), from which some of them give finite results, while others give divergent results which have no analog in the gauge side. It would be very interesting to understand if those diagrams giving infinite result cancel among themselves or the Fradkin-Tseytlin term will play a role in the cancellation of divergences. It would also be interesting to check if further α' field redefinitions are necessary. One field redefinition which one could find is $G_{MN} \to G_{MN} - K\alpha' Tr\Omega_M\Omega_N$, where K is some number, which is an analog of the redefinition in the gauge side. In that case the torsion component $T_{\alpha\beta}{}^c$ did not receive α' corrections, so it will be necessary to check if this component of the torsion receives or not corrections. There is no direct analog to the α' redefinition of $E_M{}^\alpha$ found on the gauge side, so it will be interesting to check if the gravity side computations suggest a field redefinition for this component of the supervielbein.

Having found all the α' corrections to the classical constraints, the next thing to do is to tray to relate them to those found in the literature, see Gates et al. [47] , [48] and [49] and Bonora et al. [50], [51], [52], [53] [54] in which the two groups have given answers which could not be realted among them. Recently Lechner and Tonin [55] have proposed a new set of $N = 1 D = 10$ supergravity constraints. Those authors also claim that in this new formulation of $N = 1 D = 10$ supergravity the apparently not conciliated set of constraints can be related. So, a perspective of the work presented in this thesis will also be to relate the constraints coming from the pure spinor computation with the set recently proposed.

Apêndice A

Appendix

In this appendix we present the results of the background field expansions of the terms in the pure spinor heterotic sigma model.

A.1   Background Field Expansions

From the expansion of the term $\frac{1}{2}\partial Z^M \overline{\partial} Z^N B_{NM}$

$$\frac{1}{2\pi\alpha'}\int d^2z[\frac{1}{2}\Pi^B\overline{\Pi}^A Y^C H_{CAB} + \frac{1}{4}Y^A\partial Y^B\overline{\Pi}^C H_{CBA} - \frac{1}{4}Y^A\overline{\partial}Y^B\Pi^C H_{CBA} \quad \text{(A.1)}$$

$$+\frac{1}{4}Y^A Y^B \Pi^C \overline{\Pi}^D H_{DCBA}],$$

where $H_{ABC} = (-)^{a(b+n)+(c+p)(a+b)}3E_C^P E_B^N E_A^M \partial_{[M}B_{NP]}$,

$$\partial_{[M}B_{NP]} = \frac{1}{3}(\partial_M B_{NP} + (-)^{m(n+p)}\partial_N B_{PM} + (-)^{p(m+n)}\partial_p B_{MN}) \quad \text{(A.2)}$$

and $H_{DCBA} = (-)^{B(C+D)}\nabla_B H_{DCA} - (-)^{BC}T_{DB}{}^E H_{ECA} + (-)^{D(B+C)}T_{CB}{}^E H_{EDA}$.
From the expantion of $\partial Z^M \overline{J}^I A_{MI}$

$$\frac{1}{2\pi\alpha'}\int d^2z[(\overline{J}_0^I + \overline{J}_1^I + \overline{J}_2^I)(\partial Y^A A_{AI} + \Pi^A Y^B(\partial_B A_{AI} + \widetilde{T}_{BA}{}^C A_{CI}) + \Pi^A A_{AI} \quad \text{(A.3)}$$

$$+\frac{1}{2}\partial Y^A Y^B(\partial_{[B}A_{A]I} + \widetilde{T}_{BA}{}^C A_{CI}) + \frac{1}{2}Y^A Y^B \Pi^C \widetilde{T}_{CB}{}^D(\partial_D A_{AI} + \widetilde{T}_{DA}{}^E A_{EI})$$

$$-\frac{(-)^{BC}}{2}Y^A Y^B \Pi^C \partial_B(\partial_C A_{AI} + \widetilde{T}_{CA}{}^D A_{DI})$$

From the expansion of $d_\alpha \overline{\partial} Z^M E_M^\alpha$

$$\frac{1}{2\pi\alpha'}\int d^2z[(d_{\alpha0} + \widehat{d}_\alpha)(\overline{\partial}Y^\alpha + \overline{\Pi}^B Y^C \widetilde{T}_{CB}{}^\alpha)], \quad \text{(A.4)}$$

where the terms quadratic in Y were written in (4.4) .

From the expansion of $d_\alpha \overline{J}^I W_I^\alpha$

$$\frac{1}{2\pi\alpha'}\int d^2z[(d_{\alpha 0}+\widehat{d}_\alpha)(\overline{J}_0^I+\overline{J}_1^I+\overline{J}_2^I)(\frac{1}{2}Y^B Y^C \partial_C \partial_B W_I^\alpha + Y^C \partial_C W_I^\alpha + W_I^\alpha). \quad (A.5)$$

From the expansion of $\lambda^\alpha \omega_\beta \overline{\Pi}^C \Omega_{C\alpha}{}^\beta$

$$\frac{1}{2\pi\alpha'}\int d^2z[(\widehat{\lambda}^\alpha \omega_\beta + \lambda^\alpha \widehat{\omega}_\beta + \widehat{\lambda}^\alpha \widehat{\omega}_\beta)(\frac{1}{2}\overline{\partial}Y^D Y^C(\partial_{[C}\Omega_{D]\alpha}{}^\beta + \widetilde{T}_{CD}{}^E \Omega_{E\alpha}{}^\beta) + \overline{\Pi}^C \Omega_{C\alpha}{}^\beta$$

$$(A.6)$$

$$+\frac{1}{2}Y^C Y^D \overline{\Pi}^E \widetilde{T}_{ED}{}^F(\partial_F \Omega_{C\alpha}{}^\beta + \widetilde{T}_{FC}{}^G \Omega_{G\alpha}{}^\beta) + \overline{\partial}Y^C \Omega_{C\alpha}{}^\beta + \overline{\Pi}^C Y^D(\partial_D \Omega_{C\alpha}{}^\beta + \widetilde{T}_{DC}{}^E \Omega_{E\alpha}{}^\beta)$$

$$-\frac{1}{2}(-)^{DE}Y^C Y^D \overline{\Pi}^E \partial_D(\partial_E \Omega_{C\alpha}{}^\beta + \widetilde{T}_{EC}{}^F \Omega_{F\alpha}{}^\beta))].$$

From the expansion of $\lambda^\alpha \omega_\beta \overline{J}^I U_{I\alpha}{}^\beta$

$$\frac{1}{2\pi\alpha'}\int d^2z[(\lambda^\alpha \omega_\beta + \widehat{\lambda}^\alpha \omega_\beta + \lambda^\alpha \widehat{\omega}_\beta + \widehat{\lambda}^\alpha \widehat{\omega}_\beta)(\overline{J}_0^I + \overline{J}_1^I + \overline{J}_2^I)(\frac{1}{2}Y^C Y^D \partial_D \partial_C U_{I\alpha}{}^\beta \quad (A.7)$$

$$+Y^C \partial_C U_{I\alpha}{}^\beta + U_{I\alpha}{}^\beta)].$$

References

[1] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, P. S. Howe, *"Local Supersymmetry for Spinning Particles,"* Phys.Lett **B64** 435 (1976).

[2] D. H. Friedan, *"Nonlinear Models in Two Epsilon Dimensions,"* Phys. Rev. Lett. **45** 1057, (1980); *"Nonlinear Models in Two + Epsilon Dimensions,"* Annals Phys. **163** 318, (1985).

[3] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *"The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model,"* Annals Phys. **134** (1981) 85.

[4] O. A. Bedoya, *"Invariância Conforme no Modelo Sigma da Corda,"* Disertação de Mestrado, **IFT-D.004/04** (2004).

[5] P. Ramond, *Field Theory: A Modern Primer* (Addison-Wesley Publishing Company, 1990).

[6] C. Callan e L. Thorlacius, *"Sigma Models and String Theory, In Particles, Strings and Supernovae,"* TASI **1988**, eds. A. Jevicki, C. -I. Tan. (Singapure: World Scientific, 1989).

[7] R.R Metsaev and A.A. Tseytlin, *"Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor,"* Nucl. Phys. **B293**, 385, (1987).

[8] J. Polchinski, *"String Theory. Vol 2: Superstring Theory and Beyond,"* Cabridge, (1998).

[9] M.B Green, J.H Schwarz and E.Witten, *"Superstring Theory. Vol. 1:Introduction"*, Cambridge monographs, (1987).

[10] M. Green and J. Schwarz, *"Anomaly Cancellations in supersymmetric $D = 10$ Gauge Theory and Superstring Theory,"* Phys. Lett. **B149** (1984) 117.

[11] M. Green and J. Scwarz, *"Covarinat Description of Superstring,"* Phys. Lett. **B136** (1984), 367.

[12] W. Siegel, *"Hidden local supersymmetry in the supersymmetric particle action"*, Phys. Lett. **B128** (1983), 397.

[13] E. Witten, *"String Theory Dynamics in Various Dimensions,"* Nucl. Phys. **B443** (1995) 85.

[14] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, *"Strings in Background Fields",* Nucl. Phys. **B262** (1985) 593

[15] A. Sen, *"Heterotic String in an Arbitrary Background Field,"* Phys. Rev. **D32** (1985) 2102; *"Equations of Motion for the Heterotic String Theory from the Conformal Invariance of the Sigma Model,"* Phys. Lett. **55** (1985) 1846.

[16] M.T Grisaru, P. Howe, L Mezincescu, B.E.W. Nilsson and P.K. Townsend, *"N = 2 Superstrings in a Supergravity Background,"* Phys. Lett. **B162** (1985) 116.

[17] M. T. Grisaru, H. Nishino and D. Zanon, *"β− Function Approach to the Green-Schwarz Superstring,"* Phys. Lett. **B206** (1988) 625; *"β− Functions for the Green-Schwarz Superstring,"* Nucl. Phys. **B312** (1989) 363.

[18] M.T. Grisaru and D. Zanon, *"Covariant Quantization of the Green-Schwarz Sigma-Model,"* Phys. Lett **B218** (1989) 26.

[19] N. Berkovits, "Covariant Quantization of the Green-Schwarz Superstring in a Calabi-Yau Background," Nucl. Phys. **B431** 258(1994), hep-th/9404162; N. Berkovits, emph"A New Description of the Superstring," proceedings of VIII J.A. Swieca Escola de Verão hep-th/9604123.

[20] J. de Boer and K. Skenderis, *"Covariant Computatation of the Low Energy Effective Action of the Heterotic Superstring,"* Nucl. Phys. **B481** 129 (1996) [arXiv: hep-th/9608078].

[21] D. L. Nedel, *"Derivação no super-espaço das equações efetivas da supercorda tipo II,"* Tese de Doutorado, IFT 2003.

[22] D. L. Nedel, *"Consistency of Superspace Low Energy Equations of Motion of 4D Type II Superstring with Type II Sigma Model at Tree-Level",* Phys. Lett. **B573** (2003) 217, [arXiv: hep-th/0306166]; *"Superspace Type II 4D Supergravity from Type II Superstring,"* PoS WC2004 020, 2004, [arXiv:hep-th/0412152].

[23] N. Berkovits, *"Super-Poincaré Covariant Quantization of the Superstring"*, JHEP 0004, (2000), 018, **hep-th** (0001035).

[24] N. Berkovits and P. S. Howe, *"Ten-dimensional Supergravity Constraints from the Pure Spinor Formalism for the Superstring,"* Nucl. Phys. **B635** (2002) 75 [arXiv:hep-th/0112160].

[25] O. Chandía and B. C. Vallilo, *"Conformal Invariance of the Pure Spinor Superstring in a Curved Background,"* JHEP **0404** (2004) 041; [arXiv:hep-th/0401226].

[26] O.A. Bedoya and O. Chandia, *"One-loop Conformal Invariance of the Type II Pure Spinor Superstring in a Curved Background,"* JHEP **0701** (2007) 042; [arXiv:hep-th/0609161].

[27] J. Kluson, *"Note about Classical Dynamics of Pure Spinor String on $AdS_5 \times S^5$ Background,"* Eur. Phys. J. **C50:1019** (2007) [arXiv:hep-th/0603228]. J. Kluson *"Note About Redefinition of BRST Operator for Pure Spinor String in General Background,"* arXiv:0803.4390 [hep-th].

[28] P.S. Howe, *"Pure spinor lines in superspace and ten-dimensional supersymmetric theories,"* Phys. Lett. **B258** (1991) 141

[29] N. Berkovits, *" ICTP Lectures on Covariant Quantization of the superstring",* *[arXiv] hep-th/0209059*

[30] C. R. Mafra, *"Introdução aos Formalismos de Green-Schwarz e Espinores Puros da Supercorda,"* Tese de Mestrado **IFT-D.003/05**

[31] W. Siegel, *"Classical Superstring Mechanics,"* , Nucl. Phys. **B263** (1986) 93.

[32] D. Z. Marchioro, *"Relação entre os formalismos de Green-Schwarz e espinores puros para a supercorda,"* Tese de Doutorado **IFT-T.001/05**.

[33] N. Berkovits and D.Z. Marchioro, *"Relating the Green-Schwarz and pure spinor formalism for the superstirng,"* JHEP **0501** 018 (2005), hep-th/0412198.

[34] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[35] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **0601**, 005 (2006) [arXiv:hep-th/0503197].

[36] L. Anguelova, P. A. Grassi and P. Vanhove, "Covariant one-loop amplitudes in D = 11," Nucl. Phys. B **702**, 269 (2004) [arXiv:hep-th/0408171].

[37] N. Berkovits and B.C Vallilo, *"Consistency of super-Poincaré covariant superstring tree amplitudes, "* JHEP **0007** (2000) 015 [arXiv hep-th/0004171].

[38] C. R. Mafra, *"Four-point one-loop amplitude computation in the pure spinor formalism,"* JHEP **0601**, 075 (2006) [arXiv:hep-th/0512052].

[39] N. Berkovits and C. R. Mafra, *"Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms,"* Phys. Rev. Lett. **96**, 011602 (2006) [arXiv:hep-th/0509234].

[40] N. Berkovits and C.R. Mafra, *"Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism"*, JHEP **0611** (2006) 079.

[41] N. Berkovits and V. Pershin, *"Supersymmetric Born-Infeld from the Pure Spinor Formalism of the Open Superstring,"* JHEP**0301** (2003) 023; [arXiv:hep-th/0205154].

[42] C.M Hull and E. Witten, *"Supersymmetric Sigma Models and the Heterotic String,"* Phys. Let. **B160** (1985) 398.

[43] J. J. Atick, A. Dhar and B. Ratra, *"Superspace formulation of ten-dimensional N = 1 supergravity coupled to N = 1 super Yang-Mills,"* Phys. Rev. **D 33**, 2824 (1986).

[44] G. Chapline and N. Manton *"Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions,"* Phys. Lett **120B** 105, (1983).

[45] P.S. Howe, *"Pure Sinors, function superspaces and supergravity theories in ten and eleven dimensions,"* Phys. Lett. **B273** 90, (1991).

[46] O. A. Bedoya, *"Yang-Mills Chern-Simons corrections from the Pure Spinor Superstring,"* arXiv:0807.3981[hep-th].

[47] S. Bellucci and S. J. Gates, Jr., *"D =10 , N =1 Superspace Supergravity And The Lorentz Chern Simons Form,"* Phys. Lett. **B208** 456, (1988).

[48] S. Bellucci, D.A. Depireux and S.J. Gates Jr., *"Consistent And Universal Inclusion Of The Lorentz Chern-Simons Form In D=10, N=1 Supergravity Theories,"* Phys. Lett. **B238** 315, (1990).

[49] S.J. Gates, Jr., A. Kiss and W. Merrell, *"Dynamical equations from a first-order perturbative superspace formulation of 10D, N=1 string-corrected supergravity (I),"* JHEP **0412** 047, (2004), [arXiv:hep-th/0409104].

[50] R. D'Auria, P. Frè, M. Raciti and F. Riva, *"Anomaly Free Supergravity In D=10.1. The Bianchi Identities And The Bosonic Lagrangian,"* Int. J. Mod. Phys. **A3** 953, (1988).

[51] M. Raciti, F. Riva and D. Zanon, *"Perturbativa Approach to D=10 Superspace Supergravity With a Lorentz Chern-Simons Form, "* Phys. Lett. **B227** 118, (1989).

[52] L. Bonora, P. Pasti and M. Tonin, *"Superspace Formulation Of 10-D Sugra+Sym Theory A La Green-Schwarz, "* Phys. Lett. **B188** 335, (1987).

[53] L. Bonora, M. Bregola, K. Lechner, P. Pasti and M. Tonin, *"Anomaly Free Supergravity And Superyang-Mills Theories In Ten-Dimensions, "* Nucl.Phys. **B296** 877, (1988).

[54] L. Bonora et. al., *"Some remarks on the supersymmetrization of the Lorentz Chern-Simons in D=10 N=1 supergravity theories,"* Phys. Lett. **B277** 306, (1992).

[55] K. Lechner and M. Tonin, *"Superspace formulations of ten-dimensional supergravity,"*, arXiv: 0802.3869[hep-th]

[56] J. Polchinski, *"String Theory. Vol I: An Introduction to the Bosonic STring,"* Cabridge, (1998).

[57] O. Chandía, *"A note on the Classical BRST Symmetry of the Pure Spinor String in a Curved Background,"* JHEP **0607** (2006) 019 [arXiv:hep-th/0604115].

[58] A. Sen, *"Local Gauge And Lorentz Invariance Of The Heterotic String Theory,"* Phys. Lett. **B 166** 300, (1986).

[59] C.M. Hull and P.K. Townsend, *"World-sheet Supersymmetry and Anomaly Cancellation in the Heterotic String,"* Phys. Lett. **B178** 187, (1986).